Spherical Astronomy Part II

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Recap :

* 1 hour (unit of angle) := 15°
* Rotation of earth : 1° in 4 minutes
  Revolution of earth : 1° / day $\Rightarrow$ 2 hours (angle) in 1 month
* Altitude of north celestial pole (NCP) = latitude of observer φ
* Need 2 angles to specify position on the celestial sphere
* Horizontal Coordinates: Azimuth $\rightarrow$ like compass heading A
  Altitude $\rightarrow$ up/down tilt h
  fixed to the observer
* Equatorial Coordinates: Right Ascension $\rightarrow$ like longitude, but fixed to sky (approximately) α
  Declination $\rightarrow$ projection of latitudes to the sky δ

* Celestial Equator : Projection of earth's equator onto the sky
Ecliptic : Projection of earth's orbital plane onto the sky
→ tilted at $23\frac{1}{2}°$ to the celestial equator

Cel. Equator & Ecliptic intersect at 2 points → Equinoxes
(RA = 0°, Dec = 0°) at the Vernal (Spring/March) equinox.
(RA = 180°, Dec = 0°) at the Autumnal (Fall/September) equinox.

* While the earth rotates & revolves, the RA/Dec coordinate system remains fixed in the sky (approximately)

* Meridian: Projection of observer's longitude onto the Cel. sphere
Local Sidereal Time (LST) : RA that is on the meridian
Hour angle of an object = LST − RA of the object

* 1 sidereal day ≃ 1 (mean solar) day − 4 minutes

$= 23^h\, 56^m$

This is (approximately) the time it takes for a star to appear in the same place.

The sidereal clock (LST) increases by $1.002\,737\,909\,35^s$ per second

* <u>Great circle</u>: Circle on a sphere whose radius = radius of sphere. Geodesics on a sphere

Angular perimeter $= 360^\circ$

Eg: Celestial Equator, Ecliptic

<u>Not</u> great circles: Circles of fixed declination other than cel. equator

Ref: Jean Meeus Chapter 7

Julian Day: A continuous count of (mean solar) days since the beginning of the year -4712 (in Julian calendar)

Has nothing to do with any calendar!

Algorithm to convert calendar date (Gregorian) into Julian Day number: (Formula 7.1 in Meeus 2^{nd} ed. (1998))

$D \to$ day, $M \to$ month, $Y \to$ year.

If $M < 3$, i.e. January or February, change

$$\left.\begin{aligned} M &\to M+12 \\ Y &\to Y-1 \end{aligned}\right\}$$ (i.e. treat Jan, Feb as if they were 13th and 14th month of previous year)

Let $A := \lfloor Y/100 \rfloor$ $\quad B := 2 - A + \left\lfloor \frac{A}{4} \right\rfloor$

Then $$JD = \lfloor 365.25(Y+4716) \rfloor + \lfloor 30.6001(M+1) \rfloor + D + B - 1524.5$$

safety factor to prevent float quantization issues.

This magical formula takes into account all the quirks of our calendar like February having 28 days, leap years Gregorian reform ...

Note:

$\lfloor \ \rfloor$ = floor function (removes fractional part for +ve nos)

Examples:

$\lfloor 3.2 \rfloor = 3$

$\lfloor 7 \rfloor = 7$

In C++/Python use int(x) for $\lfloor x \rfloor$ when $x > 0$

* **Note:** By tradition Julian Day begins at 12 PM UT

This formula gives the JD for 0 hours (i.e. 12 AM) UT on the date, which is why the result has fraction 0.5

Example: 1st Jan 2000, 0:00 UT

$$Y = 1999, \quad M = 13, \quad D = 1$$

$$A = \left\lfloor \frac{1999}{100} \right\rfloor = 19 \quad ; \quad B = 2 - 19 + \left\lfloor \frac{19}{4} \right\rfloor = -13$$

$$\Rightarrow \lfloor 365.25(1999 + 4716) \rfloor + \lfloor 30.6001 * (13+1) \rfloor + 1 - 13 - 1524.5$$

$$= 2451544.5$$

N.B. 12 PM UT on 1st Jan 2000 will be 2451545 ← this instant of time is called J2000.0

Calculating Local Sidereal Time (Jean Meeus Chap. 12)

1. Convert local time to UT (date + time)
2. Calculate JD for 0:00 UT on the date (i.e. fractional part 0.5)
3. Calculate:

$$T := \frac{JD - 2451545.0}{36525}$$

Julian centuries since J2000 (Formula 12.1)

4. Calculate

GMST at J2000.0

0.002737^s extra per second, accumulated over 36525 days

$$(GMST)_{00:00} = 6^h 41^m 50^s.54841 + 8640184^s.812866\,T + 0^s.093104\,T^2 - 0^s.0000062\,T^3$$

(Formula 12.2)

Greenwich mean Sidereal time at 0:00 on the date

Obscure corrections that are negligible for amateur astronomy over the next 300 yrs

This formula is valid only for 0 hrs UT on a given date...

5. Take the UT, convert it to seconds, multiply by 1.002737909350 and add it to $(GMST)_{00:00}$ to get GMST at the instant

i.e. $GMST = (GMST)_{00:00} + (UT - 00:00) \times 1.002737909350$

6. To convert GMST to LST, add 4^m per degree of longitude, i.e.

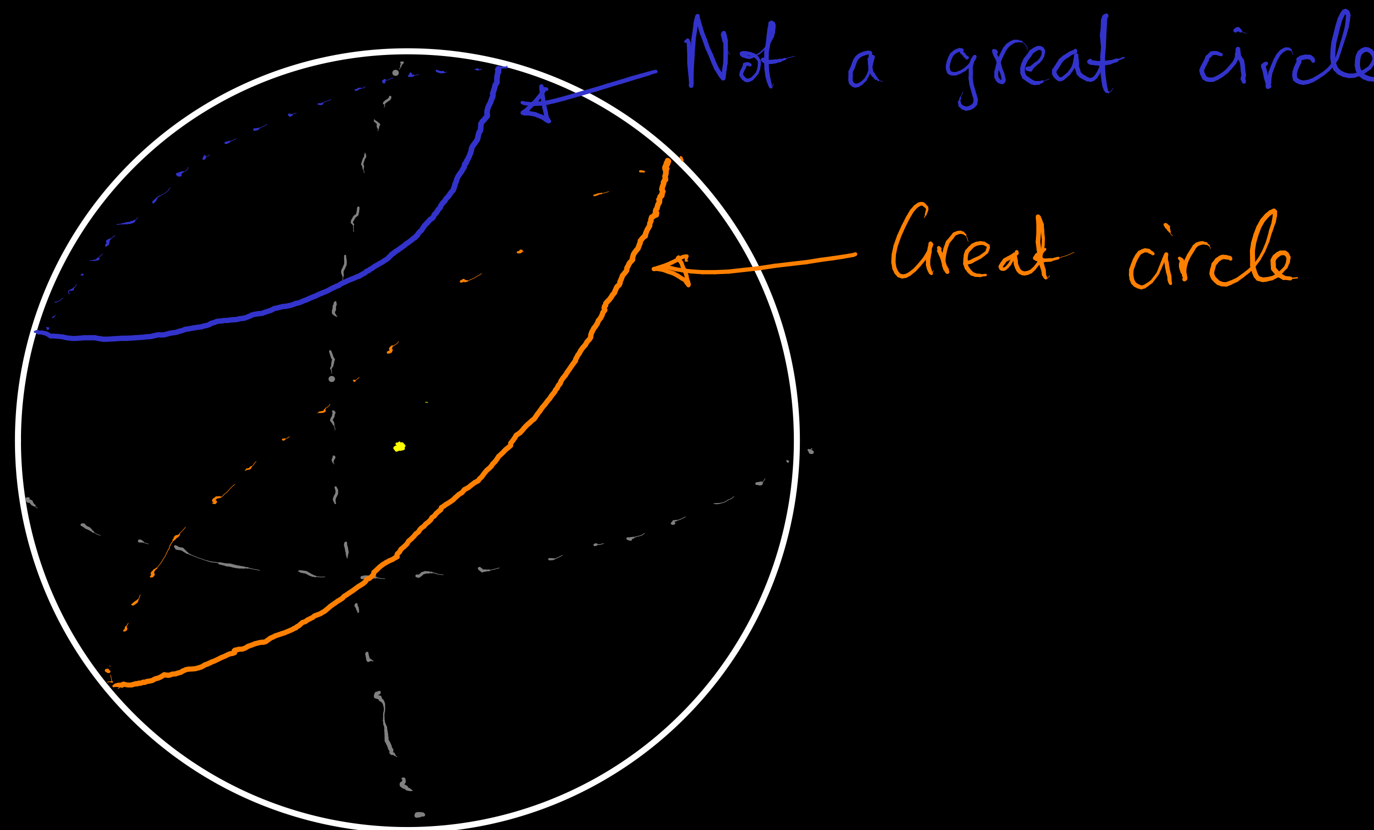
$$LST = GMST + 4^m \times (\text{longitude (in degrees)})$$

Spherical Geometry:

* The Geometry we have studied → "Euclidean Geometry"
  - lines and triangles on a plane.
* For astronomy, we need to study "lines" and triangles on the surface of a sphere.

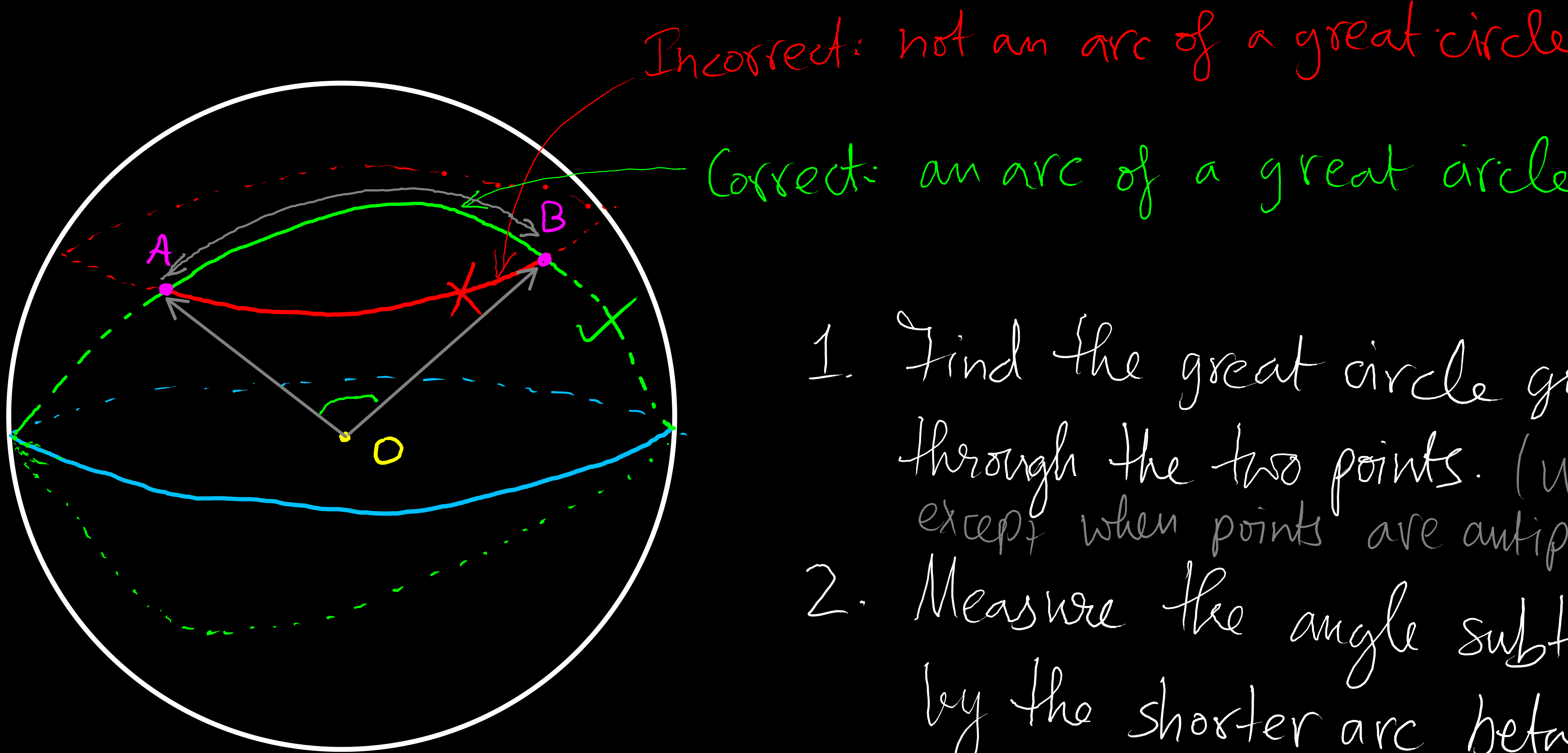
Euclidean Geometry	Spherical Geometry
Straight line	Great circle
Parallel lines don't intersect	All distinct great circles intersect at 2 diametrically opposite points
Sum of angles of a triangle $= 180^\circ$	Sum of angles of a triangle $> 180^\circ$

Reminder on great circles



1. Center of a great circle is coincident with the center of the sphere
2. Radius of a great circle = radius of sphere
3. Arclength of a great circle = 360°

Measuring distances on the sphere:



1. Find the great circle going through the two points. (unique except when points are antipodal)
2. Measure the angle subtended by the shorter arc between them at the center of the circle.

i.e. $\text{dist}(A, B) = \left(\dfrac{\text{length of arc of great circle between } A \,\&\, B}{\text{radius of the sphere}} \right)$ radians

The distance is an angular distance.

Angular circumference of any great circle $= 2\pi$ radians

$$= 360^\circ$$
$$= 24^h$$

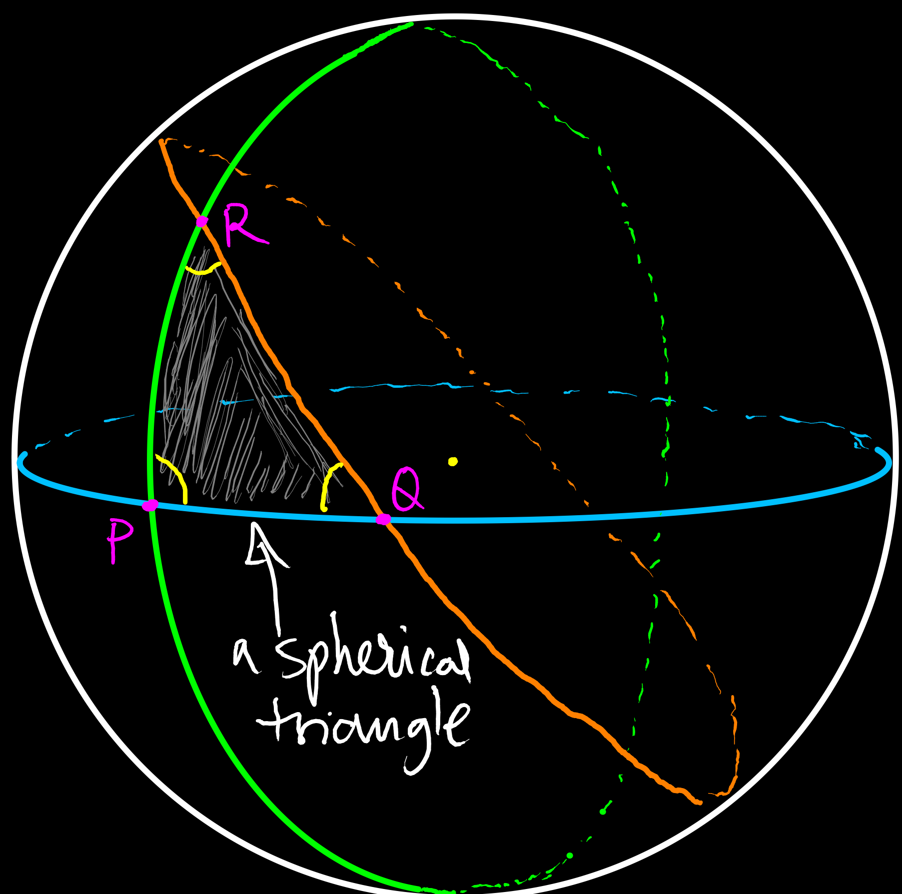
Reminder: $180^\circ = \pi$ radians, so to convert degree to radian, multiply by $\frac{\pi}{180^\circ}$. To convert radians to degrees multiply by $\frac{180^\circ}{\pi}$

This gives us a unit of (angular) length without knowing the radius of the sphere.

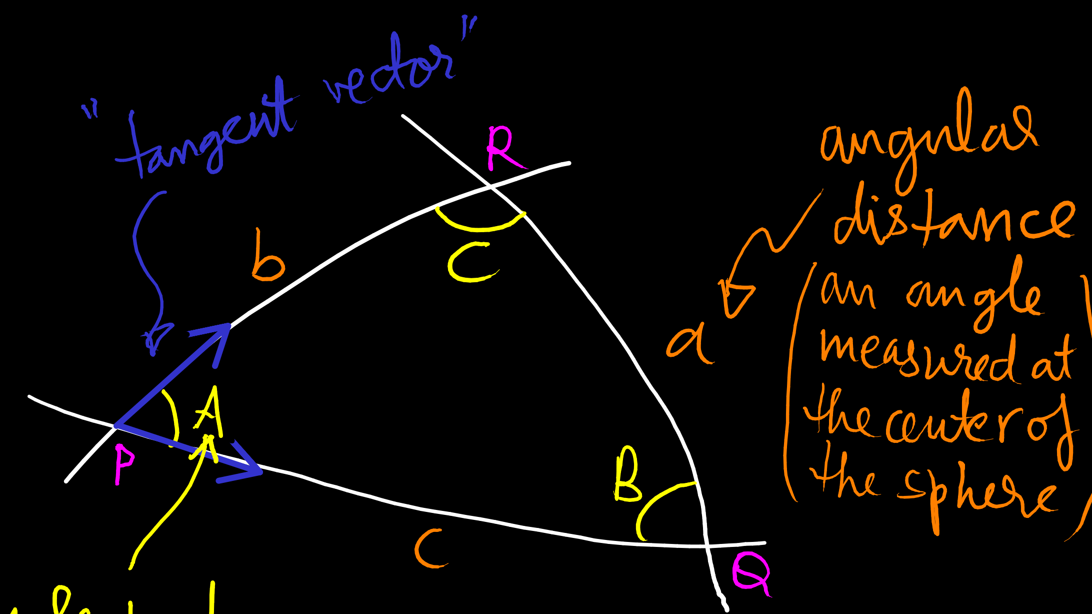
Eg: If arc between A & B has $\frac{1}{10}$ circumference of a great circle, then distance

Spherical Triangles:

Take three points, draw great circles through each pair (and take the shorter arcs)



Do not confuse the two kinds of angles: one is a (angular) distance the other is an angle.



great circles on the sphere:

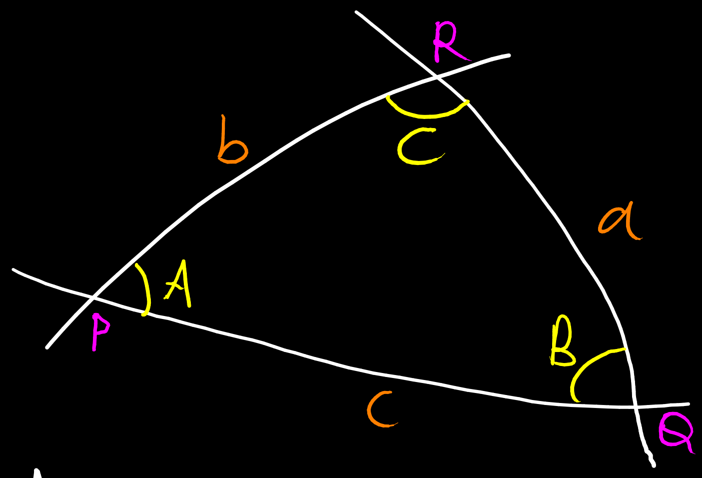
We measure this as the angle between the tangents to the great circles at the point

Convention: Side opposite the angle 'A' is 'a' and so on

Spherical Cosine Rule:

To find an unknown arc given 2 arclengths and the angle between them

$$\boxed{\cos c = \cos a \cos b + \sin a \sin b \cos C}$$



arc opp. to angle "C" ← can be found given arcs adjacent to angle C, and angle C

Similarly, $\cos a = \cos b \cos C + \sin b \sin c \cos A$

$\cos b = \cos c \cos a + \sin c \sin a \cos B$

Spherical Sine Rule:

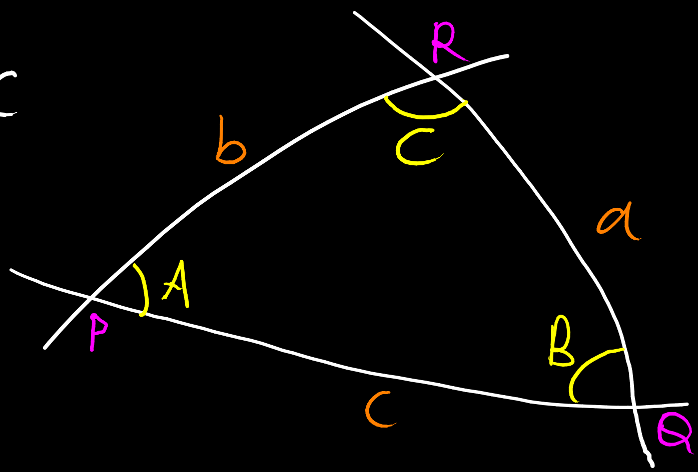
$$\boxed{\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}}$$

Second spherical cosine rule:

To find an unknown angle given the other two angles and the arc opp. it

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

and cylic permutations thereof.



"Haversine" version of spherical cosine rule:

Numerical issue: when θ is small, $\cos\theta$ hardly changes.

Eg: $\cos(1^\circ) = 0.9998473\ldots$

$\cos(2^\circ) = 0.9993908\ldots$

$\Rightarrow$ Trying to find small θ from $\cos(\theta)$ using $\cos^{-1}$ is hard.

Solution: Use $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

In spherical cosine rule,

$$\boxed{\cos c = \cos a \cos b + \sin a \sin b \cos C}$$

substitute $\cos c = 1 - 2\sin^2\left(\frac{c}{2}\right)$

Then using $\cos(A-B) = \ldots$ identity, we get

$$\sin^2\left(\frac{c}{2}\right) = \sin^2\left(\frac{a-b}{2}\right) + \sin a \sin b \sin^2\left(\frac{C}{2}\right)$$

Note: $\sin^2\left(\frac{\theta}{2}\right)$ is also known as "haversine" of θ

To calculate the arc "c":

$$c = 2\sin^{-1}\left[\sqrt{\sin^2\left(\frac{a-b}{2}\right) + \sin a \sin b \sin^2\frac{C}{2}}\right]$$

Thumb Rule: Prefer Haversine if you know the angle is between 0° and 90° and the sign is not important

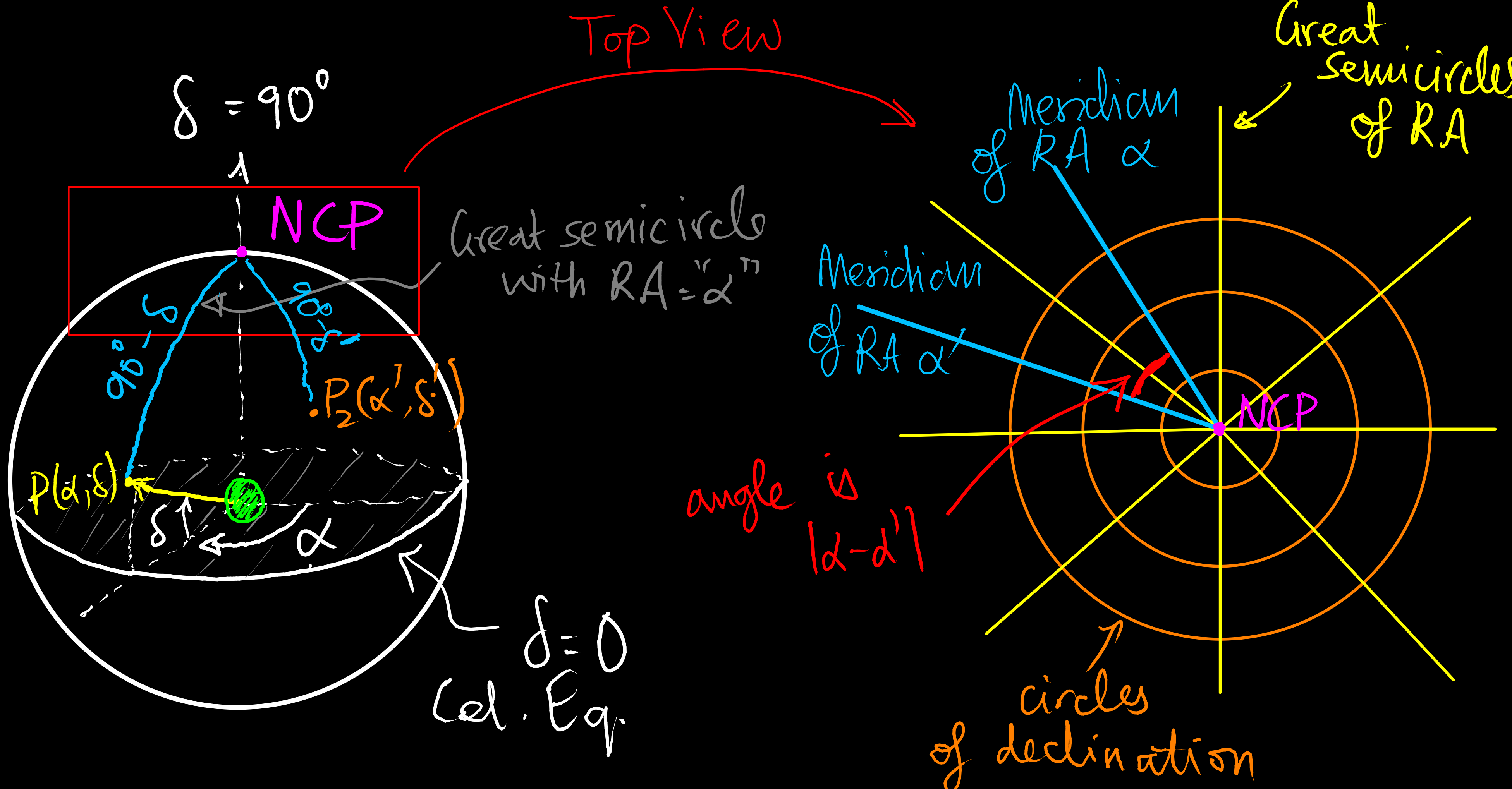
Strategy for applying these formulas:

1. Construct a triangle using the points of interest (i.e. one or more stars/objects) and special points (i.e. NCP, zenith etc.)

2. Use the great circles in your coordinate system to relate coordinate values to arclengths (eg: zenithal distance $90^\circ - h$)

3. The angle is usually 90°, one of the coordinates, or the unknown in the problem.

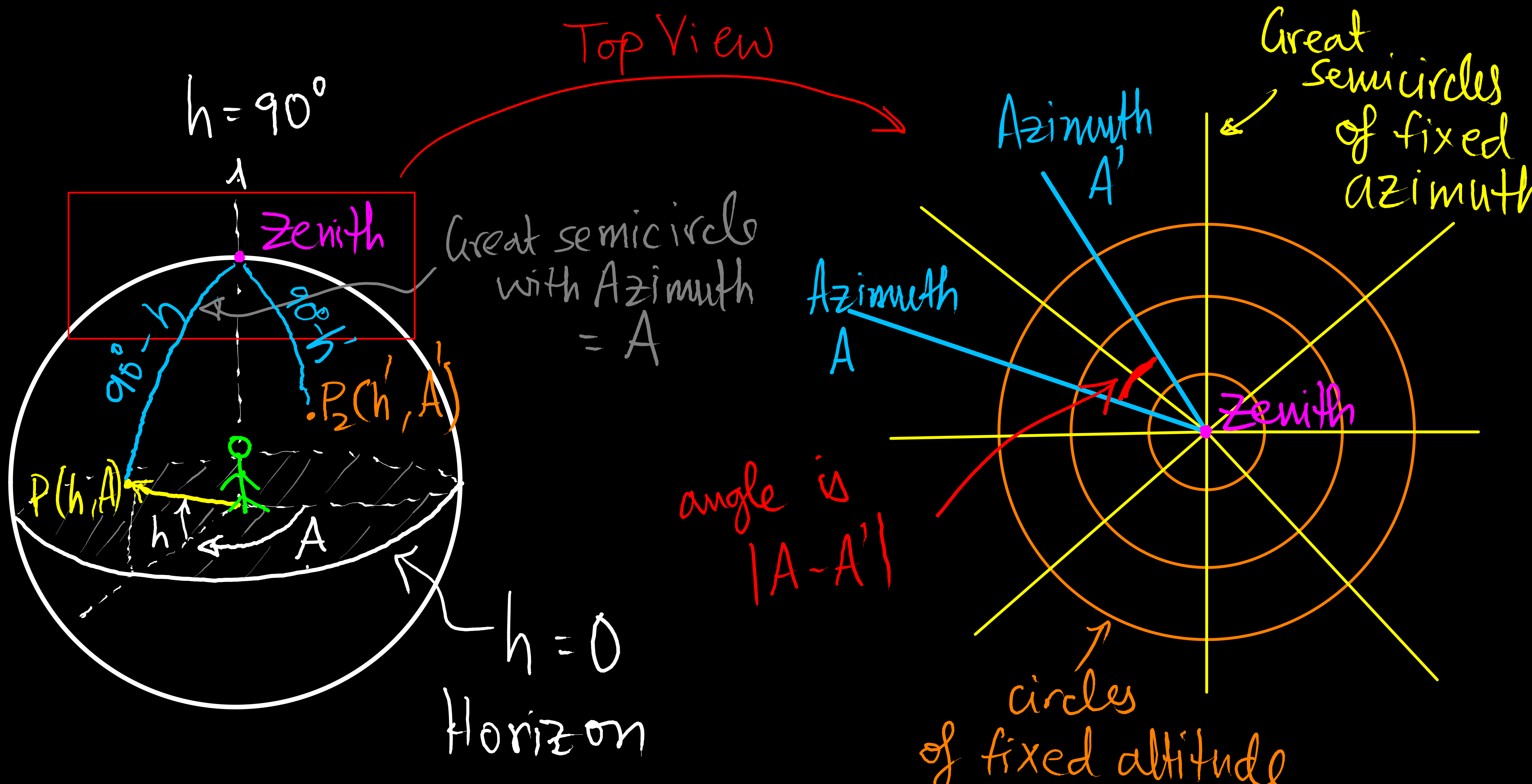
Equatorial Coordinates:



Observation 1: Arc joining NCP to a point (α, δ) has angular length $90^\circ - \delta$

Observation 2: Angle between two RA meridians at the north pole is the difference in RA

Horizontal Coordinates:

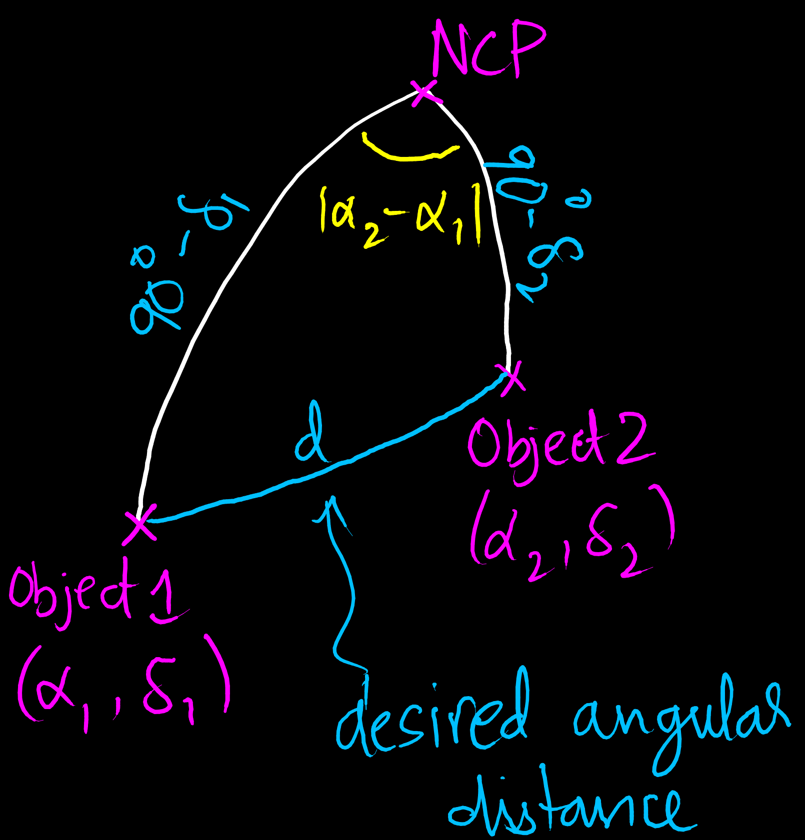


Observation 1: Arc joining zenith to a point (h, A) has angular length $90° - h$

Observation 2: Angle between azimuth semicircles at the north pole is the difference in Azimuth

Application 1: Calculate angular distance between two stars/objects given their RA/Dec.

Strategy: Construct a triangle using the two objects & NCP.



* The angle at NCP is $|\alpha_2-\alpha_1|$
* The two adjacent arcs are $(90°-\delta_1)$ and $(90°-\delta_2)$
* The unknown is the arc opp. NCP

$$\Rightarrow \cos d = \cos(90°-\delta_1)\cos(90°-\delta_2) + \sin(90°-\delta_1)\sin(90°-\delta_2)\cos|\alpha_2-\alpha_1|$$

Using $\sin(90°-\theta) = \cos\theta$, $\cos(90°-\theta) = \sin\theta$, $\cos(|\theta|) = \cos(\theta)$

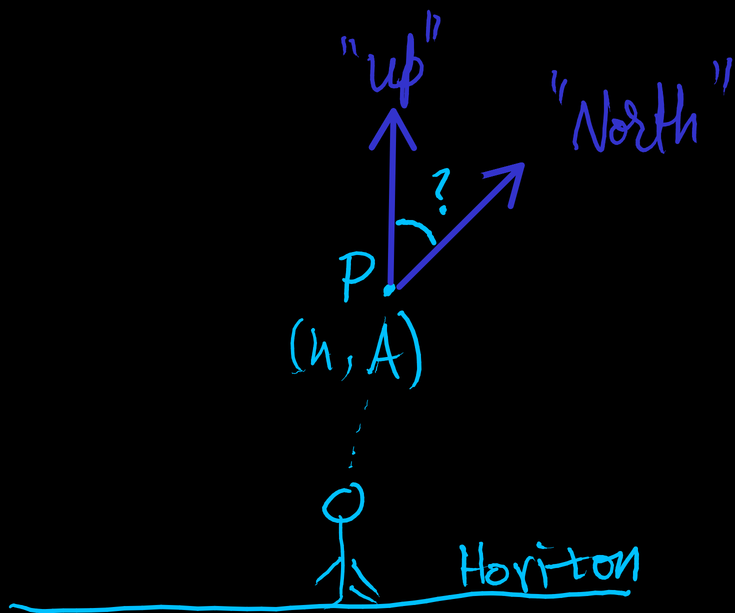
$$\Rightarrow \cos d = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_2-\alpha_1)$$

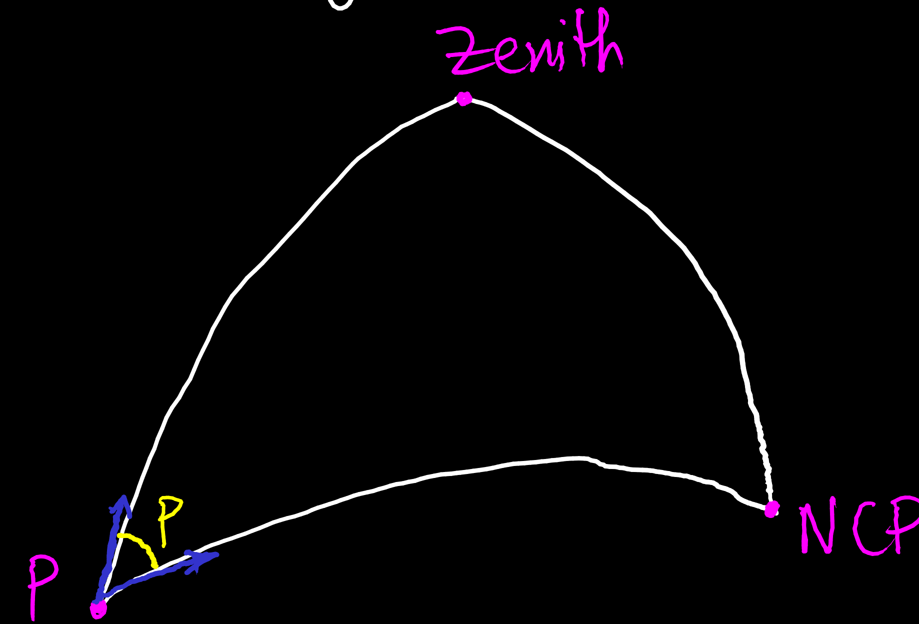
Haversine version of distance formula:

This is a good candidate for Haversine:
(distance usu. $< 90°$)

$$\Rightarrow \sin^2 \frac{d}{2} = \sin^2\left(\frac{\delta_1 - \delta_2}{2}\right) + \cos\delta_1 \cos\delta_2 \sin^2\left(\frac{\alpha_2 - \alpha_1}{2}\right)$$

Application 2: Find north angle at any point (h, A)





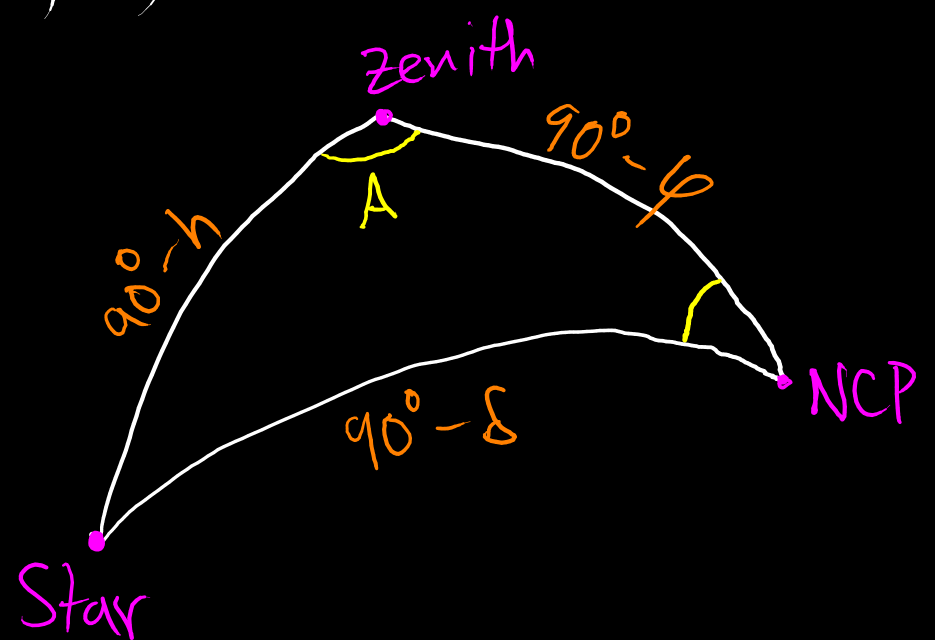
$$\cos(90^\circ - \varphi) = \cos(90^\circ - h)\cos(90^\circ - \delta) + \sin(90^\circ - h)\sin(90^\circ - \delta)\cos P$$

$$\sin\varphi = \sin h \sin\delta + \cos h \cos\delta \cos P$$

$$\Rightarrow \cos P = \frac{\sin\varphi - \sin h \sin\delta}{\cos h \cos\delta} \quad \Big| \quad \frac{\sin P}{\cos\varphi} = \frac{\sin A}{\cos\delta}$$

Application 3: The main one: Convert equatorial coords to horizontal

Given: (HA, δ) find (h, A). Recall $HA = LST - \alpha$



$$\cos(90^\circ - h) = \cos(90^\circ - \varphi)\cos(90^\circ - \delta) + \sin(90^\circ - \varphi)\sin(90^\circ - \delta)\cos(HA)$$

$$\Rightarrow \boxed{\sin h = \sin\varphi \sin\delta + \cos\varphi\cos\delta\cos HA \quad (1)}$$

This formula gives us the altitude from HA, Dec, latitude

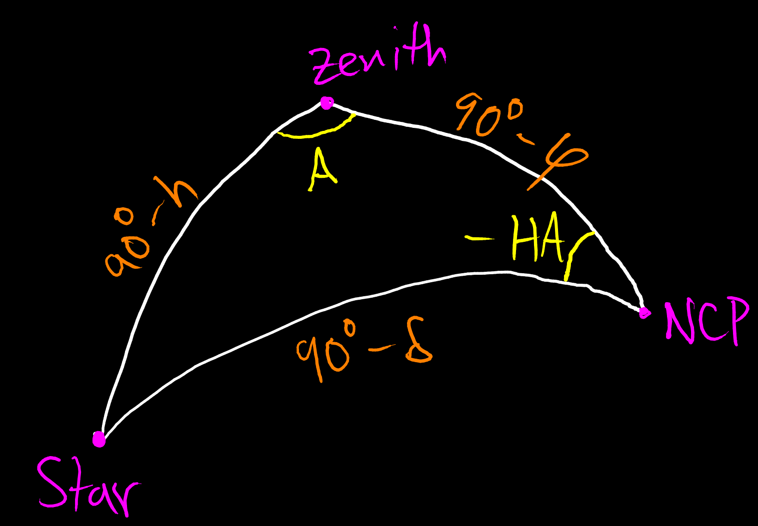
The formula for azimuth is unfortunately complicated...

$$\sin\delta = \sin h \sin\varphi + \cos h \cos\varphi \cos A \qquad (2)$$

Cosine rule for arc $90^\circ-\delta$

$$\cos\delta \sin(-HA) = \cos h \sin A \qquad (3)$$

From sine rule on the two known angles



Rearranging (2)

$$\cos h \cos\varphi \cos A = \sin\delta - \sin h \sin\varphi \qquad (4)$$

$$\cos h \sin A = -\cos\delta \sin HA \qquad (3)$$

Divide (3) by (4)

$$\tan A = \frac{-\sin HA \cos\delta \cos\varphi}{\sin\delta - \sin h \sin\varphi} \qquad (5)$$

Plug in (1) into (5) for sin(h)

$$\tan A = -\frac{\cos\varphi \cos\delta \sin HA}{\sin\delta - \sin\varphi(\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos HA)}$$

$$= -\frac{\sin HA \cancel{\cos\varphi} \cos\delta}{\sin\delta \cos^2\varphi - \sin\varphi \cancel{\cos\varphi} \cos\delta \cos HA}$$

Dividing both numerator and denominator by $\cos\delta$,

$$\boxed{\tan A = \frac{\sin HA}{\sin\varphi \cos HA - \tan\delta \cos\varphi}}$$

Note: In most computer languages, you can use atan2(,) function to compute A unambiguously:

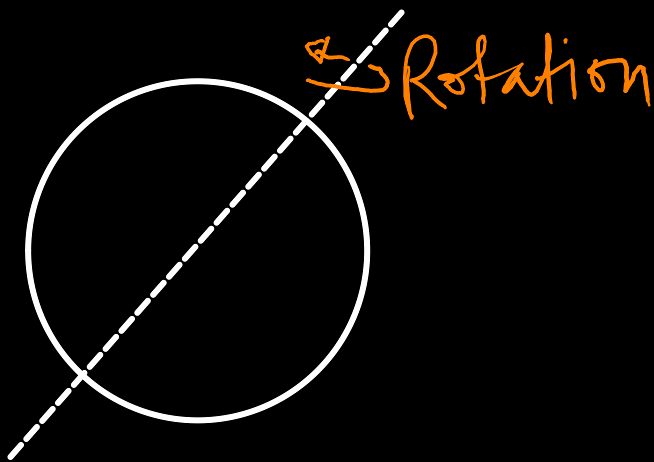
i.e. $A = \mathrm{atan2}\left(\sin HA,\ \sin\varphi \cos HA - \tan\delta\cos\varphi\right)$

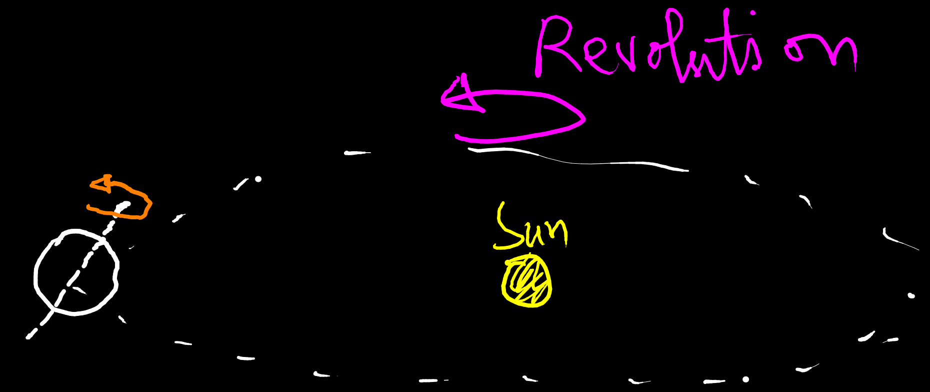
Full procedure to convert (RA, Dec) to (Alt, Az):

1. Convert from local time to UT
2. Find JD corresponding to UT date
3. Find GMST corresponding to UT date and time
4. Find LST by adding longitude offset
5. Compute: $HA = LST - \alpha$
6. Use the formulas to convert $(HA, Dec) \rightarrow (Alt, Az)$
7. Apply refraction correction if desired

Unfortunately, there is more!

~~Two~~ Three major movements of the earth:





The (RA, Dec) → (Alt, Az) conversion accounts for the above two, but not for the third

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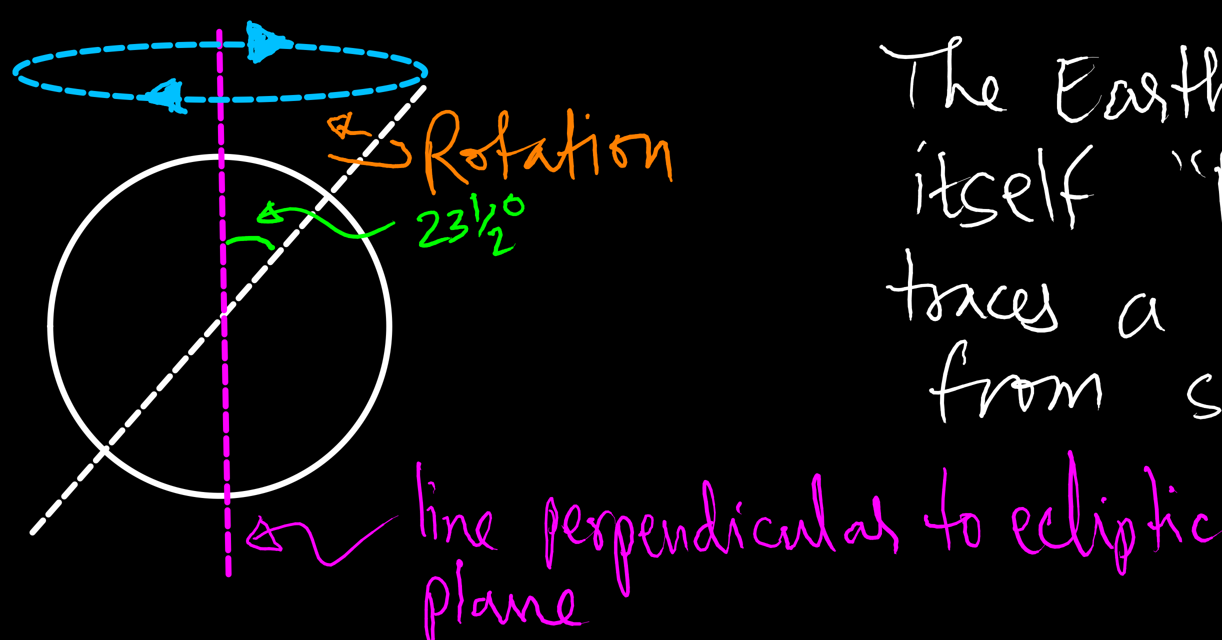
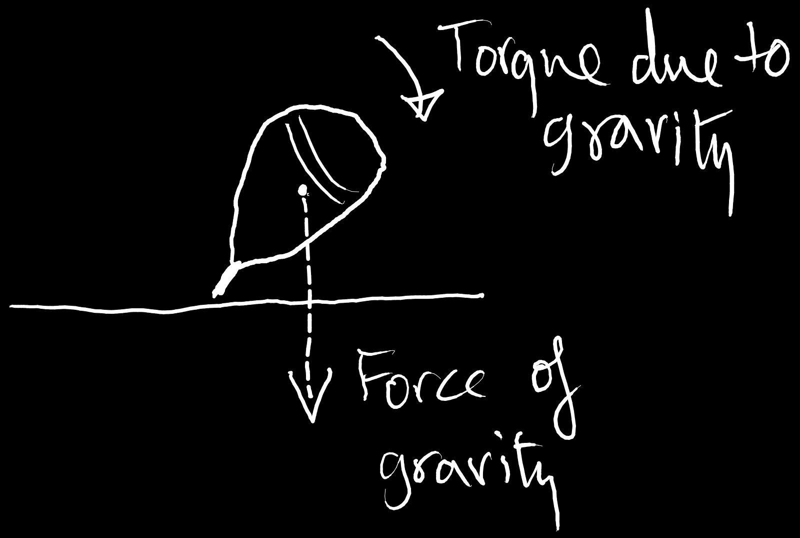
Precession:



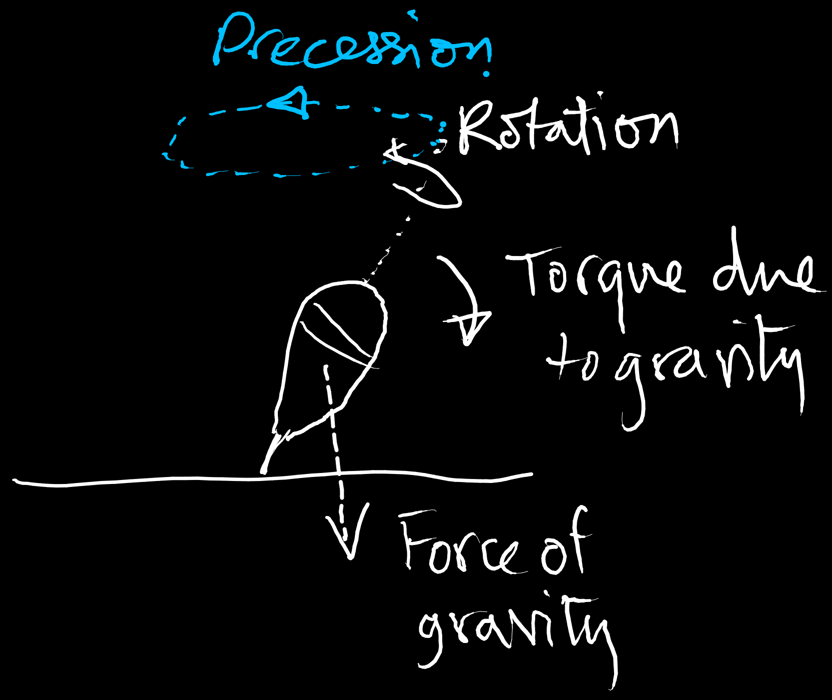
The Earth's rotation axis itself "precesses", i.e. traces a cone as seen from space

Physics of Precession:



Torque acting on stationary object: Causes it to rotate as directed by the torque

Stationary top : falls



Torque acting on already rotating object: Causes the axis of rotation to precess

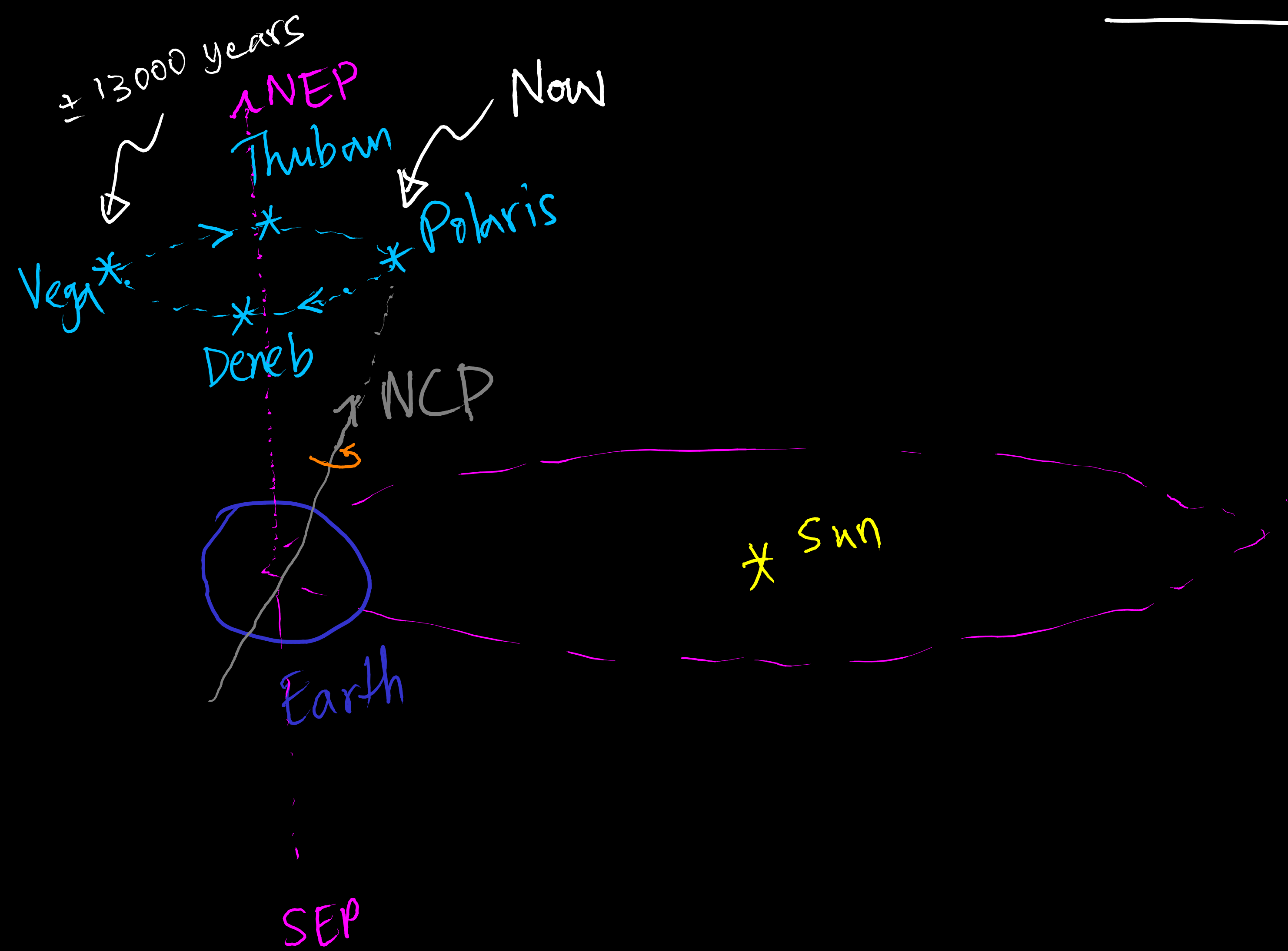
Rotating top : precesses (and eventually falls as it loses its spin)

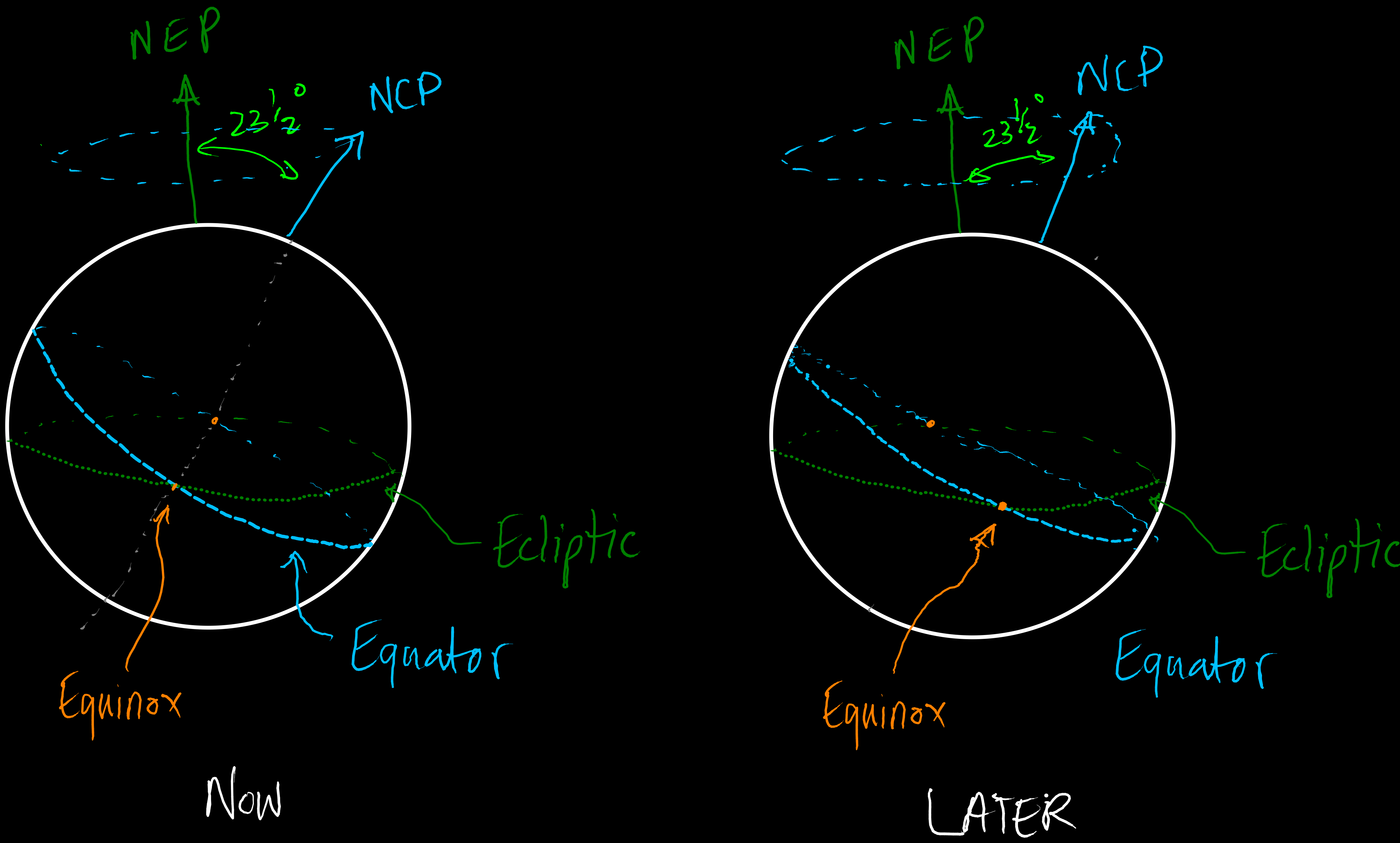
Further Reading:

"University Physics" by Young & Freedman

Precession of the equator:

* Torque from the moon and sun causes earth's axis to precess. One full orbit takes ~26000 years





* The equinoxes change
* NCP moves around the NEP, so pole stars change

* The RA/Dec coordinate grid:
    * Does not rotate with the earth
    * Does not revolve with the earth
    * It however, precesses with the earth!

* This means:
    - $\alpha = 0$ is always at Vernal Equinox (even though equinoxes move in the sky)
    - $\delta = 90^\circ$ is always at NCP (even though NCP moves in the sky)
    - RA and Dec of stars change with time (slowly) due to precession

Then what is in a catalog?

Answer: RA and Dec at a given epoch.

Eg: "J2000" Epoch := 1^{st} Jan 2000, 12 PM

ICRS: International Celestial Reference System

Practically the same as J2000.0 equatorial coords for amateur purposes

So: One more extra step needed:

Catalog coordinates (Usually J2000.0) $\longrightarrow$ Current Equatorial Coordinates ("J Now")

Precession calculation from J2000.0 to any JD.

$\alpha_0, \delta_0 \rightarrow$ J2000.0 coordinates

$\alpha, \delta \rightarrow$ coordinates at given JD

Compute: $t = \dfrac{JD - 2451545.0}{36525}$

Ref. Jean Meeus
2nd. Edition, Chap. 21
Pg 134 (formulas 21.3–21.4)

Compute:

$$\zeta = 2306''.2181t + 0''.30188t^2 + 0''.017998t^3$$

$$z = 2306''.2181t + 1''.09468t^2 + 0''.018203t^3$$

$$\theta = 2004''.3109t - 0''.42665t^2 - 0''.041833t^3$$

Compute:

$$A = \cos\delta_0 \sin(\alpha_0 + \zeta)$$

$$B = \cos\theta \cos\delta_0 \cos(\alpha_0 + \zeta) - \sin\theta \sin\delta_0$$

$$C = \sin\theta \cos\delta_0 \cos(\alpha_0 + \zeta) + \cos\theta \sin\delta_0$$

Then, $\alpha = z + \tan^{-1}\left(A/B\right)$; $\delta = \sin^{-1} C$ or $\cos^{-1}\sqrt{A^2 + B^2}$

Full procedure to convert J2000 (RA_0, Dec_0) to (Alt, Az):

1. Convert from local time to UT
2. Find JD corresponding to UT date
3. Find GMST corresponding to UT date and time
4. Find LST by adding longitude offset
5. Apply precession correction to convert (α_0, δ_0) to (α, δ)
6. Compute: $HA = LST - \alpha$
7. Use the formulas to convert $(HA, Dec) \rightarrow (Alt, Az)$
8. Apply refraction correction if desired

Other corrections we have not covered:

1. Proper motions
2. Nutation of the earth
3. Parallax (affects very few stars)
4. Stellar aberration
5. Gravitational lensing of the Sun (negligible)
6. Motion of the earth's poles
7. Figure of the earth

.... ?