Relativity and Cosmology for Amateur Astronomers

Akarsh Simha

August 2023

Special Relativity Primer

General Relativity Primer

Introduction to curved space-time

Theory of Cosmology

Counter-intuitive ideas

Our intuition is based on the length, time and mass scales we are used to in our daily lives[1]

[1]Recommended viewing: Prof. V. Balakrishnan's first lecture on "Classical Physics", available on YouTube as part of NPTEL

[2]Reference for this section: "Gravity" by James B Hartle, chapters 3 and 4

Postulates of Special Relativity

Principle of Relativity

The laws of physics take the *same form*[2] in all *inertial frames of reference*[3]
(In other words, no experiment can distinguish one inertial reference frame from another)

Constancy of the speed of light

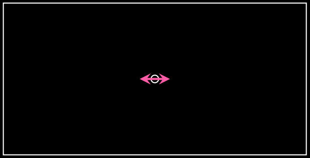
The speed of light in free-space has the same value c in all inertial frames of reference ($c = 299792458\,\mathrm{m/s}$)

[2] The mathematical form of the equation looks the same

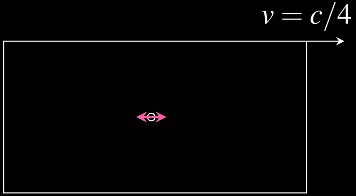
[3] An inertial frame of reference is a frame that moves with a free particle

Relativity of Simultaneity

Events that are simultaneous in one inertial frame may not be so in another



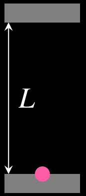
Train frame

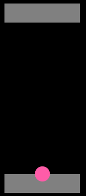


Platform frame

Time dilation

Moving clocks appear to run slow





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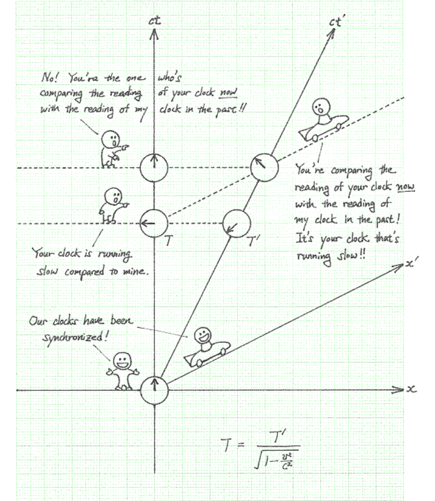

Source: Special Relativity Lecture Notes by Prof. Tatsu Takeuchi

Spacetime Interval

- We can derive that

$$\Delta s^2 := c^2 \Delta t'^2 - \Delta x'^2 = c^2 \Delta t^2 \tag{1}$$

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  Proper time is the time measured by the clock in its rest frame.
- When $\Delta s^2 < 0$, proper length is $\sqrt{-\Delta s^2}$.
  Proper length is the length of an object measured in its rest frame.

Lorentz-Fitzgerald Contraction

Moving rulers appear shorter

- Length: simultaneous measurement of the ends
- Simultaneous in one frame is not simultaneous in another!
-

$$\underbrace{\Delta x^2}_{\text{moving}} = \overbrace{\Delta x'^2 - c^2 \Delta t'^2}^{\text{rest}} < \underbrace{\Delta x'^2}_{(\text{rest length})^2} \tag{3}$$

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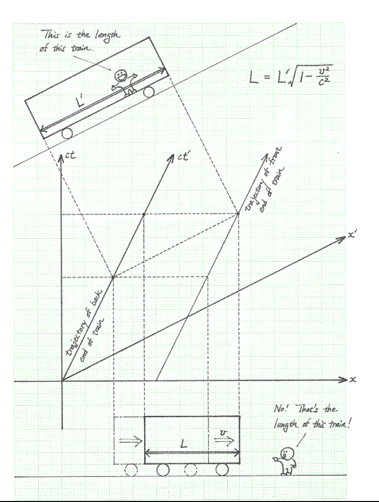

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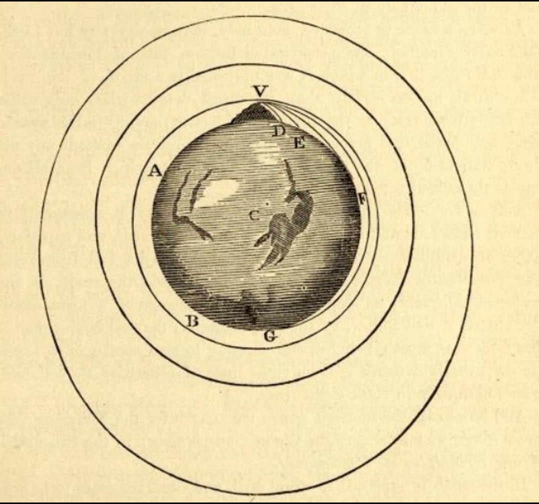
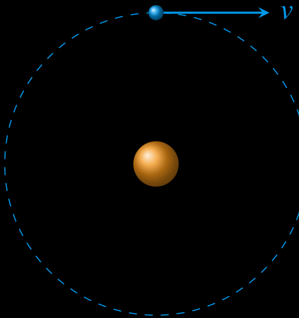
Newton's Cannonball



Orbiting planets are in free-fall

Free fall: The only force acting on the object is gravity



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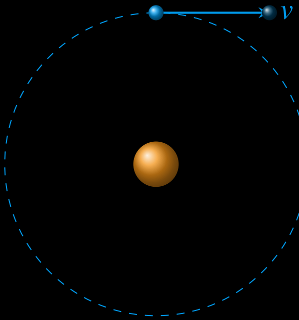

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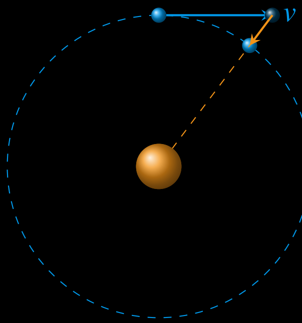

The equivalence principle

The basis for general relativity

- Gravitational Mass = Inertial Mass

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\begin{aligned}
F &= mg, \quad (g = GM/r^2) \quad \text{Newton's Gravitation Law} \\
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No experiment differentiate between a uniform gravitational field and a uniform acceleration

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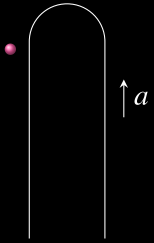

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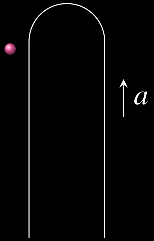

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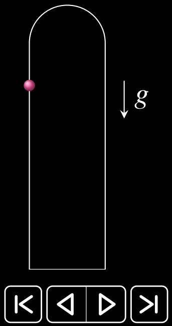

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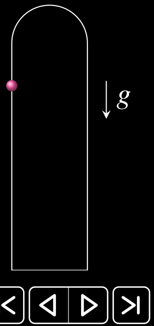
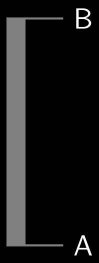


We can recover the property that "light travels in straight lines" by positing that the space has been curved by gravity.

Gravitational Redshift

Photons climbing out of a gravitational potential well undergo redshift



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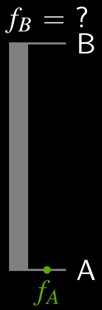

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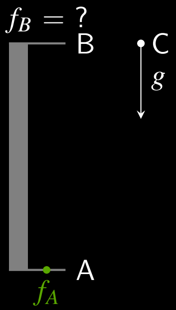

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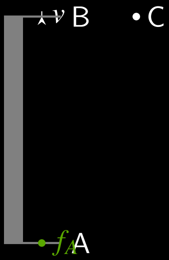

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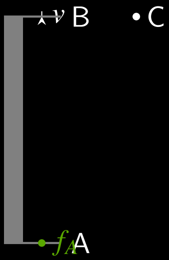

Because B is moving away from the photon as seen by C, f_B must be redshifted[4] relative to frequency seen by C $= f_A$

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Gravitational Time Dilation

- Imagine clocks at A and B ticking at frequency f_A
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- Therefore at B, 10 ticks of the clock at A as signaled by the light beam register more than 10 ticks of the clock at B.
- Thus the clock at A is running slower.

Gravitational Time Dilation

Clocks lower in gravitational fields run slower
Time runs slower lower in a gravitational field

Recap: Equivalence Principle

The Equivalence Principle

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Inertial observer in GR

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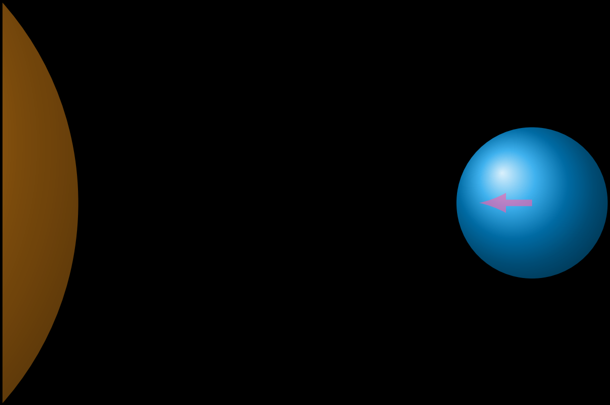
What about non-uniform gravitational fields?

Tidal Forces

Tidal forces: Differential gravity



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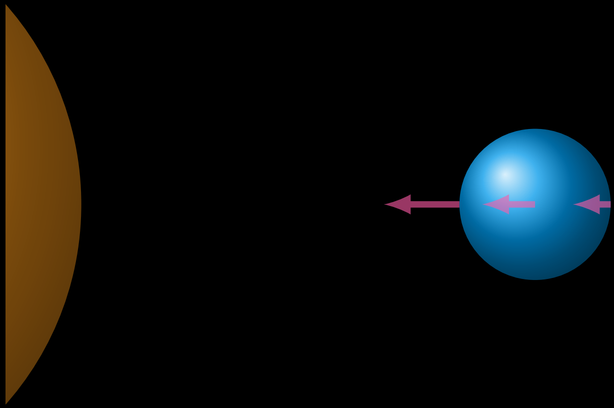

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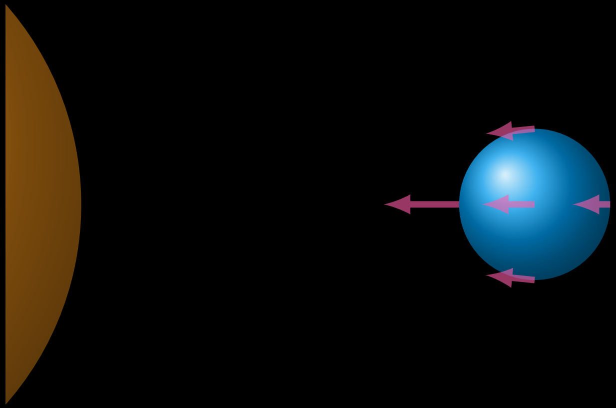

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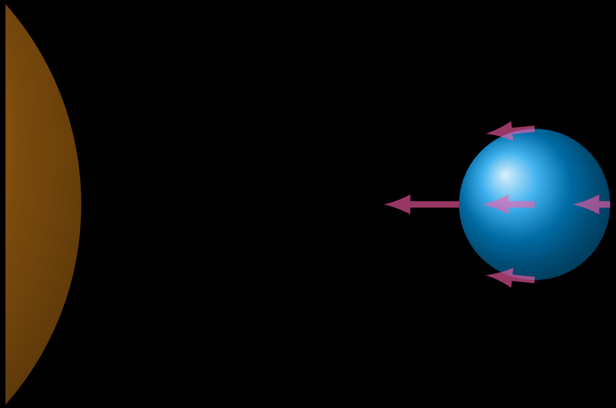
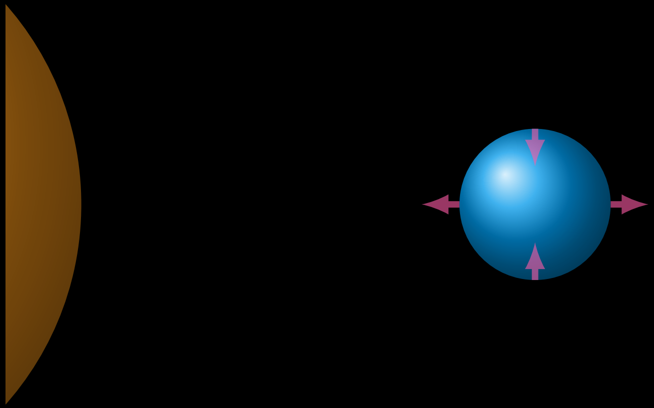


We now switch to the freely falling frame of the COG of the earth

Tidal Forces

Tidal forces: in the local inertial frame of the COG of the earth



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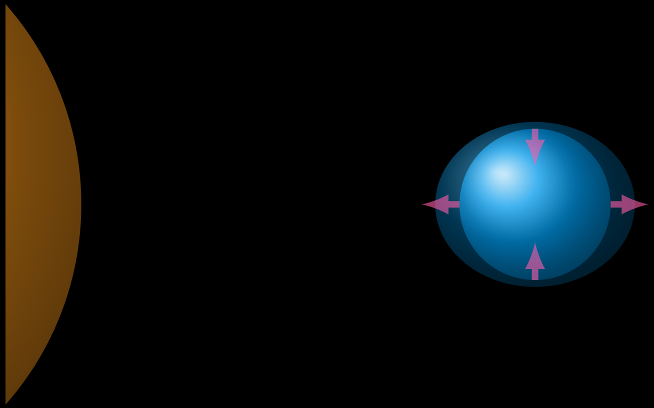

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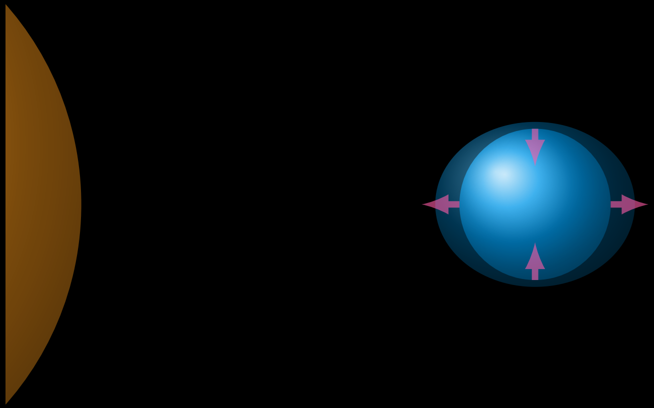
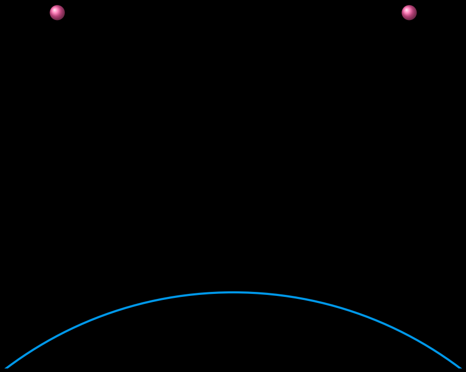


For more details, refer to Schutz's *Gravity from the ground up*

Tidal Forces

Tidal Forces

Tidal forces are the remnants of gravity for a freely falling observer



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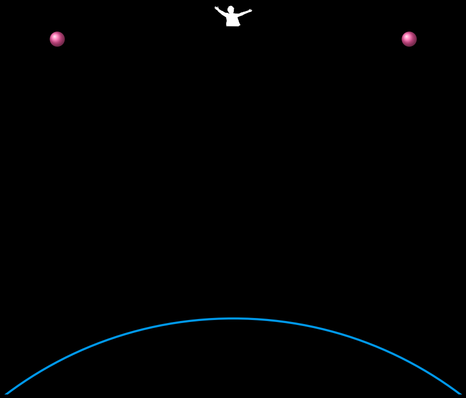

Local Inertial Frames

When the gravity is non-uniform, we can only define *local* inertial frames

Local inertial observer in GR

- Freely falling
- No other forces (other than gravity)
- Tidal effects still seen, *only local effects of gravity vanish*

The Equivalence Principle (Local Version)

Experiments in a *sufficiently small* freely-falling laboratory, over a *sufficiently short time* give the same results as the same experiments in an inertial frame in empty space

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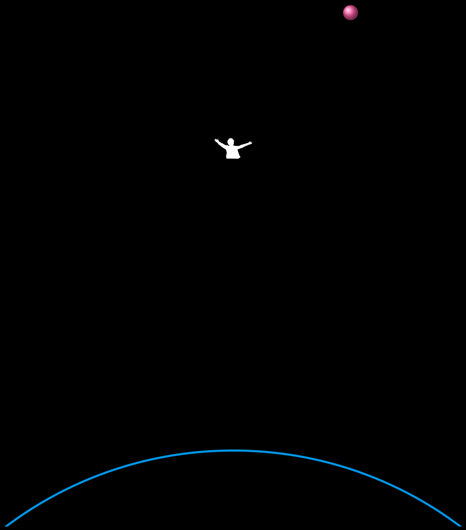

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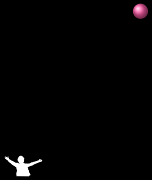

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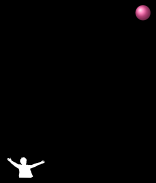
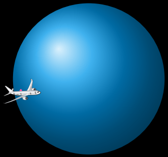


But the ball has no forces acting on it! So it must go in a straight line!

Geodesics

Straight lines in curved space



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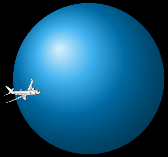
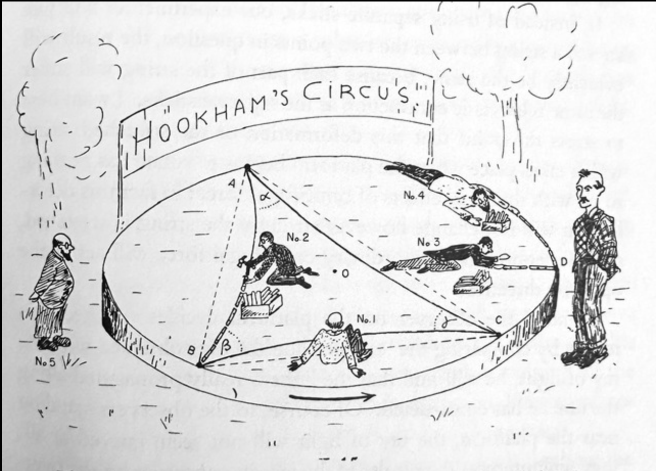


Geodesics on a sphere are great circles



The scientists were measuring something on a rotating platform*

Gamow's Rotating Platform from Mr. Tompkins[5]

[5]See also his Scientific American article from March 1961

Euclidean Geometry

- This is geometry on a flat plane or in a flat space
- What we learn in high school

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- The sum of angles of a triangle is 180°
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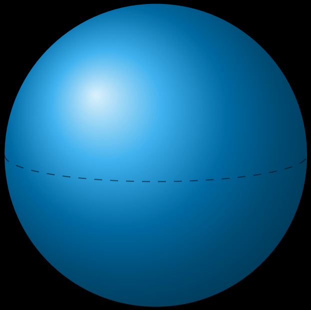
- What we learn in high school
- Parallel straight lines do not intersect
- The sum of angles of a triangle is 180°
- Ratio of circumference to diameter of a circle is π
- Distance between two points is given by Pythagoras' theorem:

$$(\text{distance})^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Example of Curvature: Surface of a Sphere

Note: We are talking about a 2-dimensional surface.
We have *embedded* it in 3D (flat) space to understand it.



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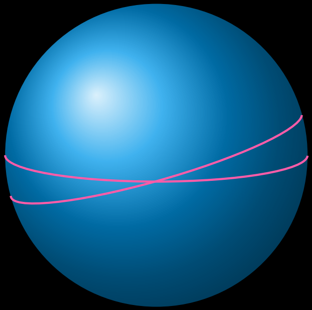

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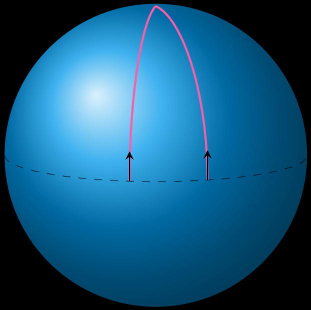

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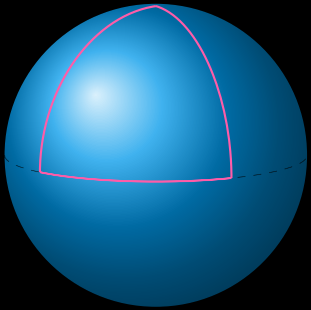

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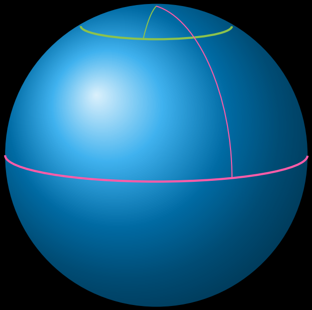

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- Ratio of circumference to diameter of a circle is always $\leq \pi$

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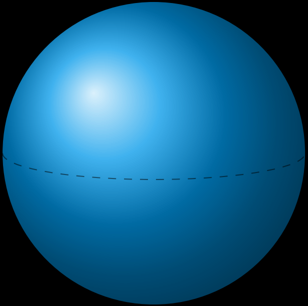
A little bit of mathematics



- Distance between two *nearby* points is given by

$$(\text{distance})^2 = \Delta\phi^2 + \cos^2\phi\,\Delta\lambda^2$$

where ϕ is latitude, λ is longitude[6].

---

[4] By convention, this is written in terms of co-latitude θ and using ϕ for longitude, e.g. in Schutz. Then this formula appears as $\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

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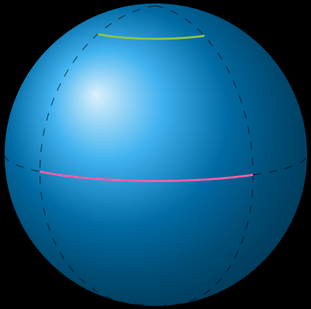

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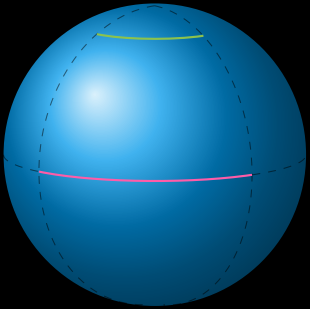

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- Note that the distance metric varies with latitude!
- That's why

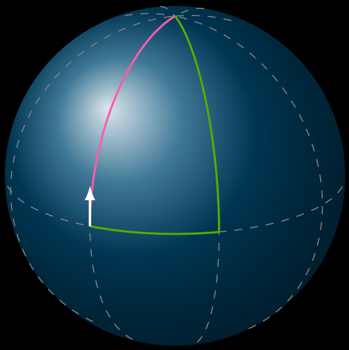
  $$\mathrm{d}s^2 = \mathrm{d}\phi^2 + \cos^2\phi\,\mathrm{d}\lambda^2.$$

  Here d$\square$ denotes a small change in $\square$.

---

[4] By convention, this is written in terms of co-latitude θ and using ϕ for longitude, e.g. in Schutz. Then this formula appears as $\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

Curvature and Parallel Transport



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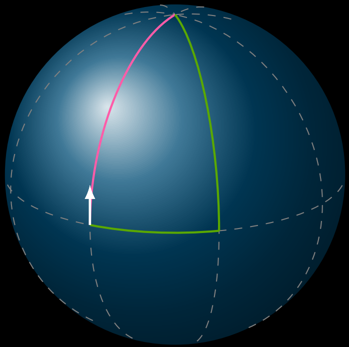

Getting different results when transporting vectors is a hallmark of curvature

Beware of embeddings

They can be misleading

- Take a flat sheet of paper and draw some geometric objects on it (e.g. a triangle)

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Curvature in Riemannian Geometry

You cannot "cure" the curvature of a truly curved surface by bending it smoothly:
You will need to "cut it open" to flatten it out.

Warning!

Mathematics ahead

The Metric

Distance formula in curved spacetime

- We've been seeing formulas like this:

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \quad \text{(Euclidean)} \\
\\
\mathrm{d}s^2 &= \mathrm{d}\phi^2 + \cos^2\phi \, \mathrm{d}\lambda^2 \quad \text{(Sphere)} \\
\mathrm{d}s^2 &= c^2\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \quad \text{(Spacetime)}
\end{aligned}
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- They all have the pattern:

$$\begin{aligned}(\text{distance})^2 &= (\text{something1})\,\mathrm{d}(\text{coord1})^2 \\ &\quad +(\text{something2})\,\mathrm{d}(\text{coord2})^2 \\ &\quad \ldots\end{aligned}$$

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- This is called the *metric*
- It tells us how to calculate distance given the coordinates of two points (space) or events (spacetime)

The Metric

Distance formula in curved space(time)

Actually, metric can in general have "cross" terms like $\mathrm{d}x\,\mathrm{d}y$, e.g.

Most general 2D metric

$$\mathrm{d}s^2 = A(x,y)\,\mathrm{d}x^2 + B(x,y)\,\mathrm{d}y^2 + C(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

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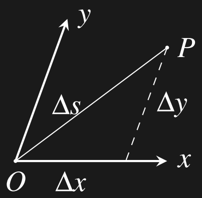
Example: Axes that are not perpendicular



Cosine rule:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 - 2\,\Delta x\,\Delta y\cos\theta$$

The Metric

The metric contains almost all information about curvature of space and time

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FLRW Metric: Metric describing the universe

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Euclidean 3-D space in Cartesian coordinates

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Euclidean 3-D space in polar coordinates

$$ds^2 = dr^2 + r^2 \left(d\phi^2 + \cos^2\phi \, d\lambda^2\right)$$

There is nothing special about one set of coordinates!

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Non-Euclidean 2-D surface of a sphere

$$\mathrm{d}s^2 = \left(\mathrm{d}\phi^2 + \cos^2\phi \; \mathrm{d}\lambda^2\right)$$

Curvature won't go away even if we use different coordinates!

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The Metric

Mathematics of the equivalence principle

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  (Second derivatives of metric $\Longrightarrow$ curvature)
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$$\mathrm{d}s^2 = \text{(some complicated stuff)}$$

Making a coordinate transform, change t to t', x to x', y to y' etc...

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- This is the mathematical analog of the equivalence principle! We can find a local inertial frame where the laws of physics apply just like in flat spacetime
- Curvature, i.e. tidal effects, do not vanish!

Coordinates

Coordinates are just a systematic way of labeling events in spacetime

Same geometry can be written in different coordinates.

Distance between two points P and Q:

$$L = \int_P^Q \mathrm{d}s$$

[7] For details, refer e.g. to Hartle's book

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Free Particle in Curved Spacetime

$$\underbrace{\frac{\mathrm{d}^2 x}{\mathrm{d}^2 \tau}}_{\text{accel}} + \Gamma^x_{tt} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau} \right)^2 = 0$$

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Geodesics

The shortest / longest distance / proper-time between two points

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The second term is the pseudo-force we call gravity!

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Metric of Newtonian Gravity

How gravity warps time near a mass

Recall: Gravity slows down clocks

[8]See Investigations 2.2 and 18.1 in Schutz's Gravity from the Ground Up

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Choose coordinate t to be the time ticked by a clock infinitely far away from the mass. Then, we know that the clock at height h in the gravitational field ticks slower[8]:

$$\delta\tau^2 = \delta t^2 \left(1 + \frac{2gh}{c^2}\right)$$

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Therefore identifying gravitational potential $\Phi = gh$, we can guess

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

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Gravitational Time Dilation

- Imagine clocks at A and B ticking at frequency f_A
- Let's synchronize A and the light of frequency f_A being emitted at A so we get a crest every time the clock ticks

Metric of Weak Gravity

But space also gets curved: remember Gamow's platform?

A more thorough derivation shows:

Static, weak-field metric

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right) (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

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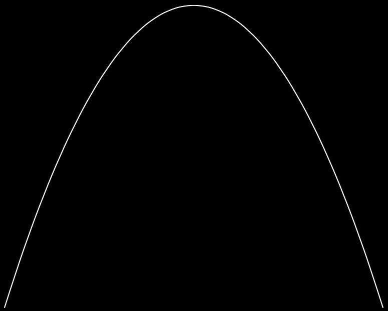
If we write out the geodesic equation for this metric, we will recover Newtonian gravity as $F_{\mathrm{grav}} = -\nabla\Phi$.

Projectile Motion Revisited

Explanation seen in a Kip Thorne lecture



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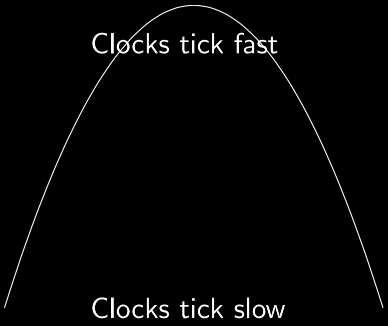

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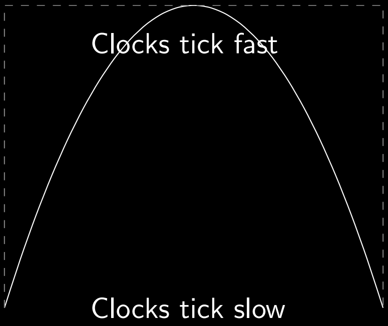

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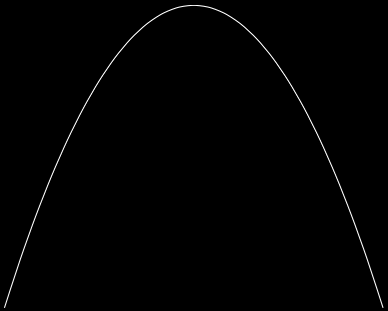

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- The *metric* characterizes a curved spacetime.
- Beware of *embedding*: extrinsic geometry is different from *intrinsic* geometry. GR is concerned with intrinsic geometry.

The Metric

A metric *typically* looks like

$$\mathrm{d}s^2 = (\cdots)c^2\mathrm{d}t^2 - (\cdots)\mathrm{d}x^2 - (\cdots)\mathrm{d}y^2 - (\cdots)\mathrm{d}z^2$$

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$\mathrm{d}s^2$ is the (small) distance between two neighboring events, whose coordinates are given by (t, x, y, z) and $(t+\mathrm{d}t, x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$.

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Most details of geometry of spacetime can be determined from the metric.

GR Concepts

- Matter tells spacetime how to curve, curved spacetime tells matter how to move
- Einstein's equations (which we did not discuss) when solved give the metric for a given matter configuration.

GR Concepts

- At any point in spacetime, one can *locally* make spacetime appear flat. An observer for which this is true is called a *local inertial observer* or a *freely-falling observer*.
- Freely falling observers follow *geodesics*, the generalization of straight lines to curved spacetime.
- A freely falling observer sees spacetime as flat around them. Thus, they do not feel gravity.
- But the effect is *local*. At further distances from the observer, there is the remaining effect of geodesic deviation or *tidal force*.
- Tidal forces are the "true" forces of gravity, seen by freely falling observers

What is Cosmology?

The study of the history of the observable universe and its evolution through physics

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- The application of physics to the study of the universe at very large scales and its history

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- Current *Standard Model* of Cosmology: Called the ΛCDM model

The Cosmological Principle

On large scales, spacetime is *isotropic* and *homogeneous*

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- *Isotropic* means "same in every direction"
- *Homogeneous* means "same at every point"

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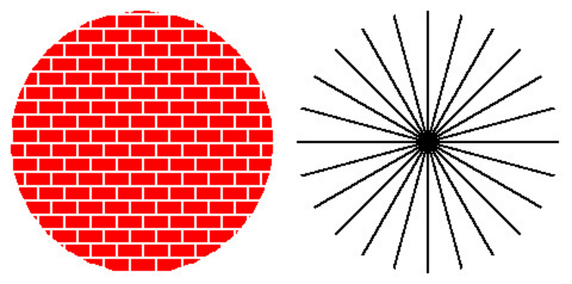
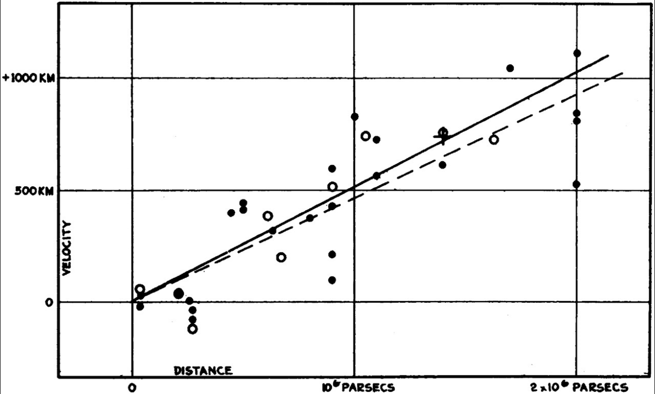


Homogeneity vs. Isotropy, Picture from Ned Wright's Cosmology Tutorial

The Universe is Expanding



Plot taken from Edwin Hubble's 1929 paper.

Curvature of Space

Homogeneity + Isotropy $\implies$ Only three possibilities for curvature of *space*

- 3D Sphere ("Spherical")
- 3D Saddle / Hyperboloid ("Hyperbolic")
- Flat ("Euclidean")

Observations show our universe is *spatially* flat to the best of our measurements.

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Note that at any given time instant, space is flat; spacetime is not.

Hubble-Lemaître Law

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v is the "velocity" of a distant galaxy, r is the "distance" to the galaxy, H is the "Hubble constant".

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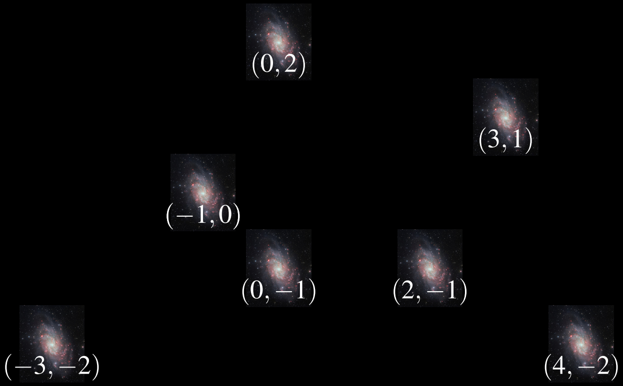
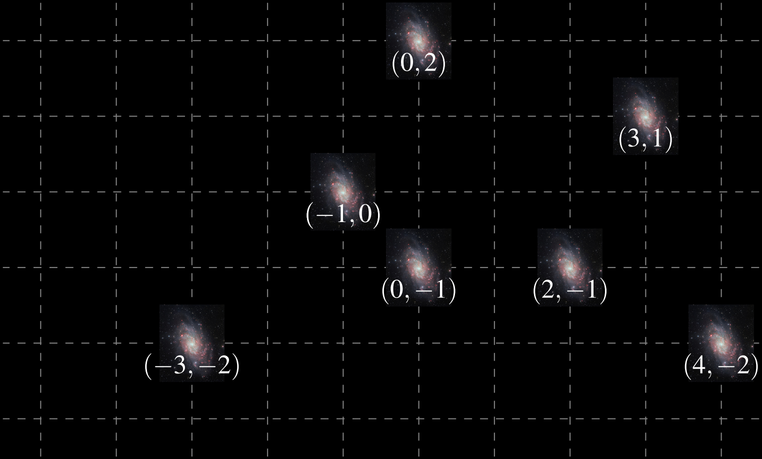
Note that H can (and does) vary with time!
"Hubble Parameter" $H(t)$.
Current value is denoted H_0, $H_0 \approx 67$–73km/s/Mpc.

Model for the expanding universe



No center of expansion: each galaxy moving away from the rest

Model for the expanding universe



Assign each galaxy a *fixed* set of *coordinates* (x, y, z).

The standard metric of the universe

Friedmann-Lemaître-Robertson-Walker (FLRW) metric, for *flat space*:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

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$a(t)$ denotes a time-varying scaling of the *coordinates* to get the *distance*.
$a(t)$ is called the *scale factor*.

Hubble Law from FLRW

Let one galaxy be at $(x, 0, 0)$ and another at $(0, 0, 0)$. Distance between them at time t is:

$$r(t) = a(t)x$$

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Some time, Δt later, the time is $t + \Delta t$, so the distance is

$$r(t + \Delta t) = a(t + \Delta t)x$$

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Some time, Δt later, the time is $t + \Delta t$, so the distance is

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Velocity is rate of change of distance:

$$v = \frac{r(t + \Delta t) - r(t)}{\Delta t} = \frac{a(t + \Delta t) - a(t)}{\Delta t}x$$

Hubble Law from FLRW

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Note x cancels out, this is the idea of Hubble-Lemître Law

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Introduce the notation $\dot{a}(t)$ for rate of change of $a(t)$,

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

This is the relationship between Hubble parameter and scale factor

Cosmological Redshift

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$z > 1$ is okay. Velocity $> c$ makes no sense. Even the special relativistic formula should not be used. In fact, relative velocity between two different points in curved spacetime makes no sense.

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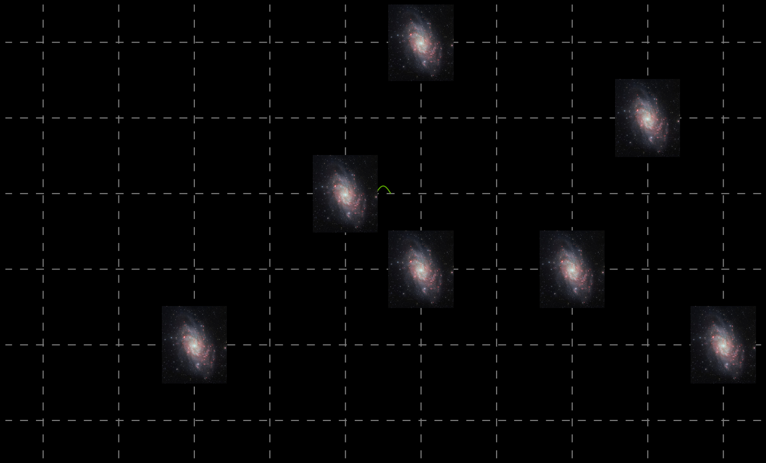
Astronomers however continue to convert z to v as a matter of convention. One should not be alarmed by $v > c$ because v is not properly a velocity.

Cosmological Redshift



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z is more properly defined through:

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Rearranging the above, we get

$$1 + z = \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}}$$

Redshifts, more properly

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but we can only observe the *past* light from galaxies, not their present positions.

Redshift! Redshift! Redshift!

It's one thing we can measure very well

- One of the most accurate measurements we can make is redshift.
- Distance can be measured through standard candles up to a certain distance, especially Type Ia Supernovae, but even this calibration is model-dependent.

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- Distance can be measured through standard candles up to a certain distance, especially Type Ia Supernovae, but even this calibration is model-dependent.
- Think of redshift z as an independent measure of distance
- Think of redshift z as a replacement of time, since higher z is looking back into the past

Contents of the Universe

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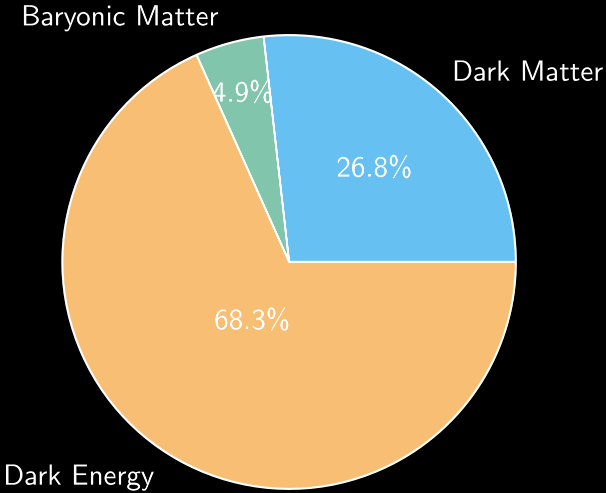
- Dark Energy (energy of empty space itself)

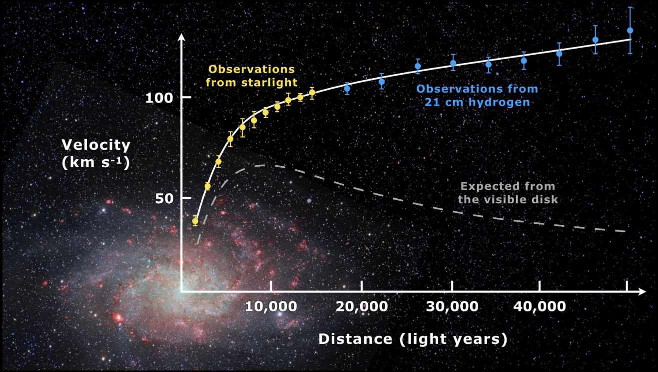
Current proportion of contents



Radiation and Neutrinos: $\Omega_R = 10^{-4}$, negligible

Dark Matter

Rotation curves of galaxies suggest matter extends way beyond their physical extent



Rotation Curve of M33 from Wikipedia, Credit: Mario De Leo.

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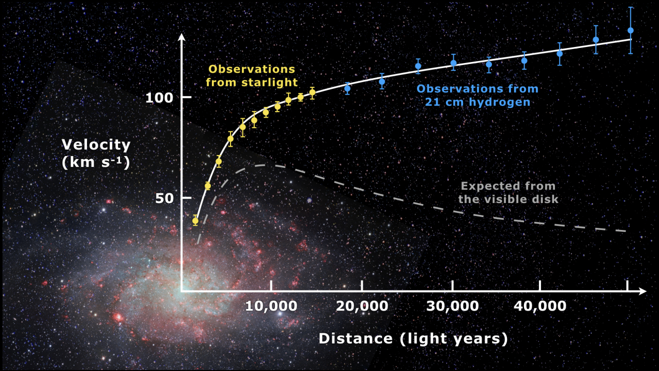

Rotation Curve of M33 from Wikipedia, Credit: Mario De Leo.

Other possible explanation: Modification to Newton's II Law (MOND), less popular.

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- Cannot be regular matter that's cold / does not emit, because it would block the light!
- Currently most anticipated: "Cold Dark Matter":
- i.e. massive particles (not light like neutrinos),
- interacts very weakly with ordinary matter and radiation
- Search for dark matter candidates still underway

Dark Energy (Λ)

What the hell is it?

Denoted by Λ. It is an energy whose density does not diminish as the universe expands.

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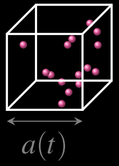
But it is needed to account for accelerated expansion of the universe.

As the universe expands...

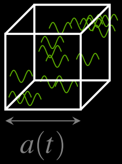
density of matter (dark and baryonic) dilutes like $a^3(t)$



Number of galaxies remains constant, but volume of region increases like $a(t)^3$.

As the universe expands...

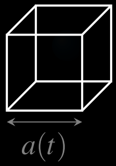
density of radiation (and neutrinos) dilutes like $a^4(t)$



Not only the density of photons decreases like $1/a(t)^3$, the energy of each photon also decreases as $1/a(t)$

As the universe expands...

density of dark energy does not dilute at all



Density of the contents of the universe

as a function of scale factor $a(t)$

Recall that $1+z = a(\text{now})/a(\text{then})$. If today's densities are:

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So the total density at any point in the past is

$$\rho(z) = \rho_M(1+z)^3 + \rho_R(1+z)^4 + \rho_\Lambda$$

Friedmann Equation

Einstein's equations roughly have the form:

$$\text{Curvature of Spacetime} = \frac{8\pi G}{c^2}\rho$$

where ρ is the density of all forms of matter and energy.
"Curvature of spacetime" can be calculated in terms of metric.

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Applying this to FLRW metric (for flat space), we get an equation of the form

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho.$$

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Note that $8\pi G/3 = 5.59 \times 10^{-10} kgm^3/s^2$ is just a constant. If we allow for curvature of space, it takes the form:

$$H(t)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a(t)^2},$$

where $k = +1$ for spherical, $k = -1$ for hyperbolic, $k = 0$ for flat space.
This is called the *Friedmann equation*.

Critical Density

$$H(t)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a(t)^2},$$

When $k = 0$, i.e. flat space, the density ρ must be equal to

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and divide the Friedmann equation by it

$$\left(\frac{H(t)}{H_0}\right)^2 = \frac{\rho(t)}{\rho_c} - \frac{kc^2}{\rho_c a(t)^2},$$

Put in the ingredients into ρ

Write in terms of z rather than t (both measure how far into the past we are looking).
Recall,

$$\rho(z) = \rho_M(1+z)^3 + \rho_R(1+z)^4 + \rho_\Lambda$$

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So we can write,

$$\left(\frac{H(z)}{H_0}\right)^2 = \frac{\rho_M}{\rho_c}(1+z)^3 + \frac{\rho_R}{\rho_c}(1+z)^4 + \frac{\rho_\Lambda}{\rho_c} - \frac{kc^2}{\rho_c a(\text{now})^2}(1+z)^2,$$

Density Parameters

We now define the density parameters as:

$$
\begin{aligned}
\Omega_M &= \frac{\rho_M}{\rho_c} \\
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So we can write,

$$\frac{H(z)^2}{H_0^2} = \Omega_\Lambda + \Omega_K(1+z)^2 + \Omega_M(1+z)^3 + \Omega_R(1+z)^4$$

The Friedmann Equation from Newtonian Physics

(Must however include all forms of energy under mass density)

Pick an origin. Then by Isotropy + Newton's Shell Theorem, galaxy of mass m at radius r from the origin flying away with velocity v has energy

$$\frac{1}{2}mv^2 - \frac{4\pi G m \rho r^2}{3} = E = \text{constant.}$$

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For escape velocity, $E = 0$, so

$$H^2(t) = \frac{8\pi G}{3}\rho_c \quad \text{(Critical Denisty)}$$

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For non-zero E, define $k := -2E/(mc^2)$,

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Also if the universe is spherically curved, big crunch is possible

The ΛCDM model

Λ: Dark Energy

CDM: Abbreviation for Cold Dark Matter

$\Omega_M = 0.316$, $\Omega_\Lambda = 0.683$, Ω_R and Ω_K negligible.

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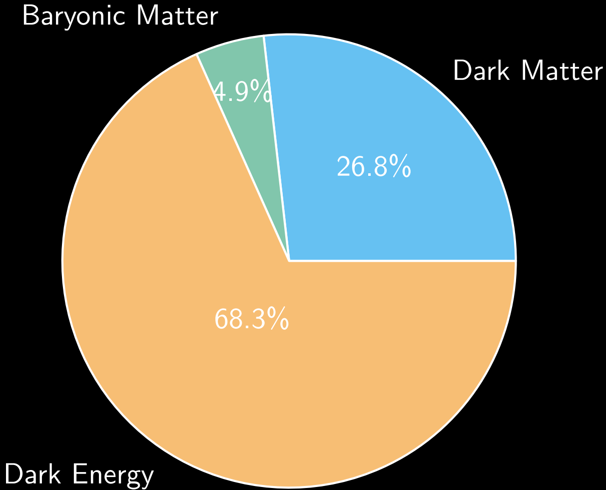
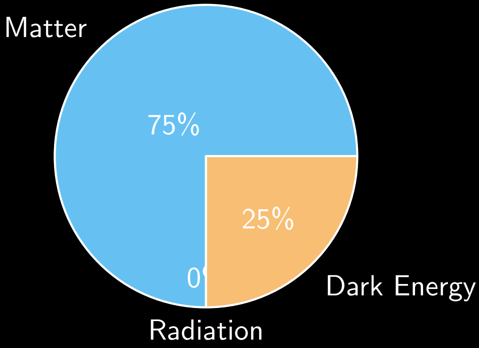


From the present to the past

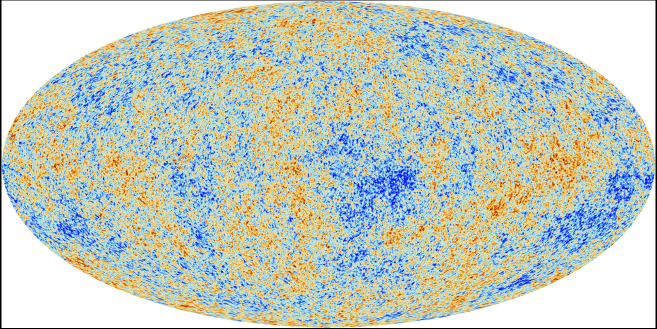
Using $\Omega_M = 0.268$, $\Omega_R = 9 \times 10^{-5}$, $\Omega_L = 0.732$



$z = 1$

Cosmic Microwave Background Radiation

The universe is filled with microwave radiation



CMBR map from Planck collaboration.

(c) ESA and Planck Collaboration

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The temperature redshifts,

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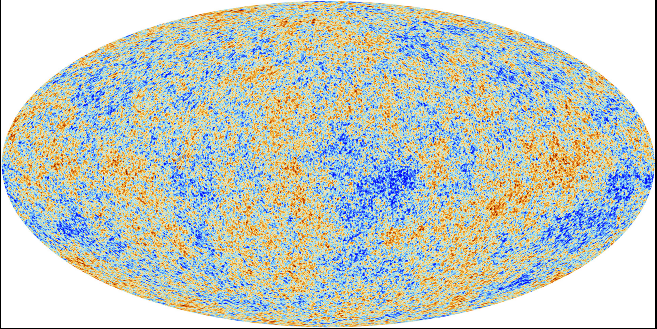
$$T(z) = 3000K, T_0 = 2.7K \implies z_{\text{recomb}} \approx 1100$$

Anisotropy of the CMBR

The temperature varies to a part in 10^5



CMBR map from Planck collaboration.
(c) ESA and Planck Collaboration

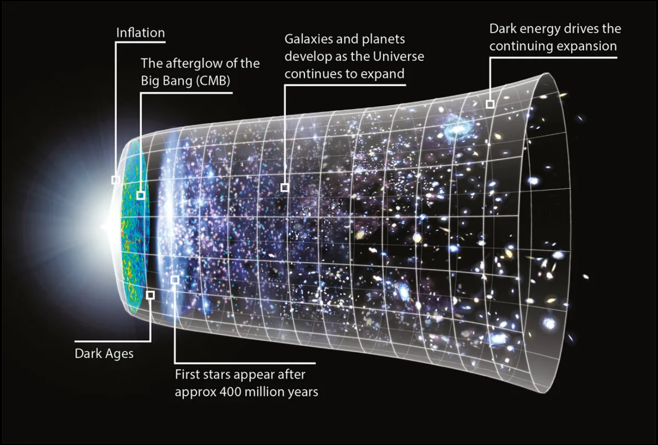
- The fluctuations in CMBR are imprints of fluctuations of matter density
- These fluctuations condensed under gravity to form galaxies

Behavior of the scale factor

Solutions of the Friedmann equation for various constituents

Dominating Constitutent	How it Dilutes	Behavior of Scale Factor
Radiation	$\rho(t) \propto 1/a(t)^3$	$a(t) \propto t^{1/2}$
Matter	$\rho(t) \propto 1/a(t)^4$	$a(t) \propto t^{2/3}$
Dark Energy	$\rho(t) = \text{constant}$	$a(t) \propto \exp(H_0 t)$

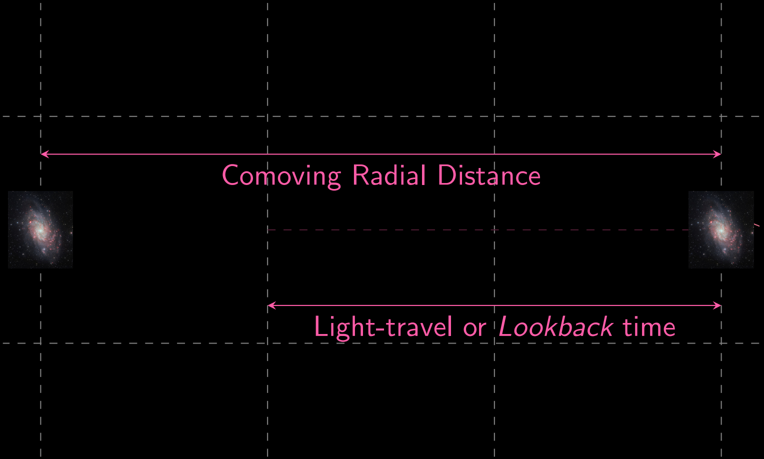
Expansion History of the Universe



Plot of $a(t)$ vs t, Credit: NASA

Different notions of distance

Different notions of distance



Comoving Radial Distance

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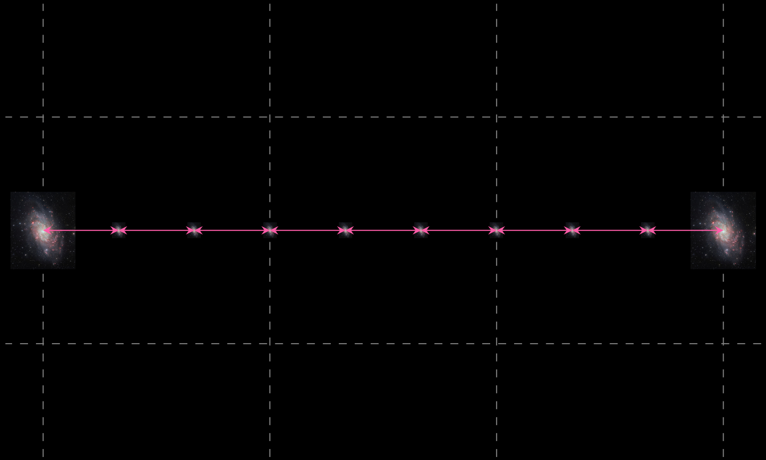
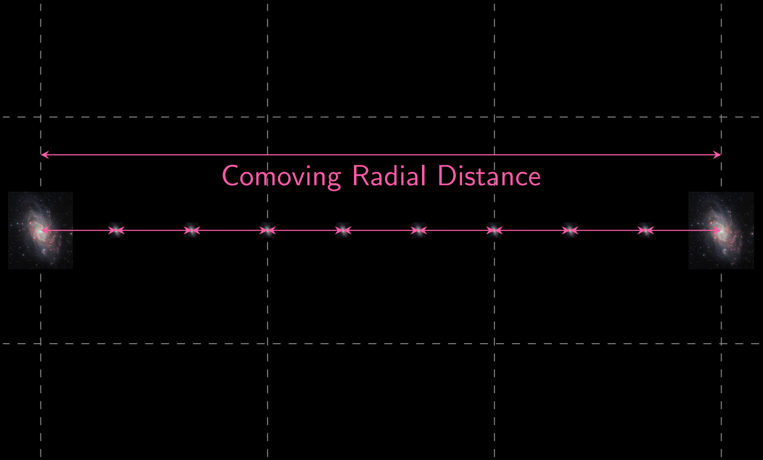
Freeze time

Comoving Radial Distance



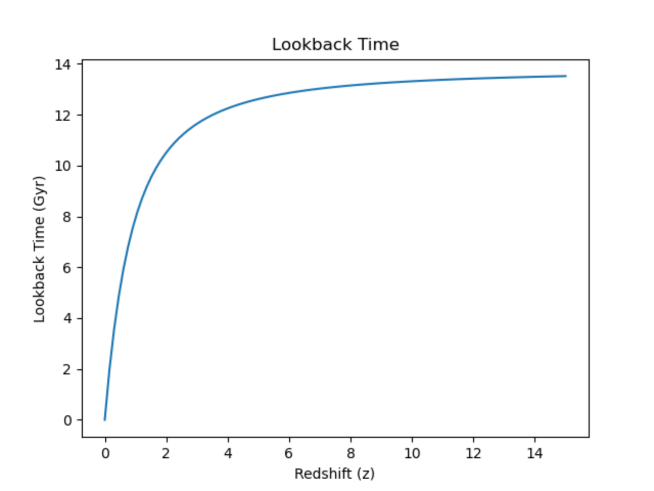
Add up the distances

Comoving Radial Distance



Add up the distances

Light Travel Time vs Redshift



Angular Size Distance

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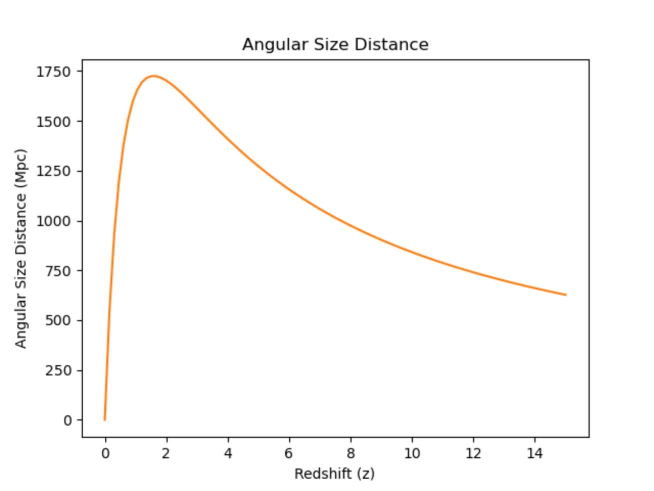
At some point, we are looking so far back, that the galaxies start getting closer to us and the angular size distance actually decreases!

Angular Size Distance



Luminosity Distance

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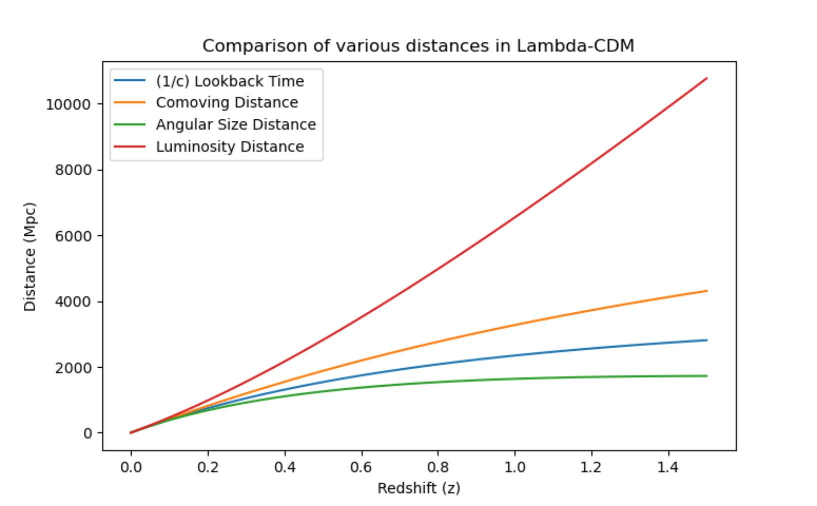
But photons redshifted, so energy reduces by $1+z$. Rate of arrival of photons also reduces by $1+z$.
Thus, the expanding-universe version is,

$$F = \frac{L}{4\pi d(z)^2(1+z)^2},$$

so the *luminosity distance* is given by $d(z)(1+z)$.

Different notions of distance



Define

$$E(z) := \Omega_\Lambda + \Omega_K(1+z)^2 + \Omega_M(1+z)^3 + \Omega_R(1+z)^4.$$

Light Travel Time:

$$t(z) := \frac{1}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')\sqrt{E(z')}}$$

Comoving Radial Distance:

$$d(z) := \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{E(z')}}$$

Different notions of distance

Flat Universe

Angular Size Distance:

$$d_A(z) := \frac{d(z)}{1+z}$$

Luminosity Distance:

$$d_L(z) := d(z)(1+z)$$

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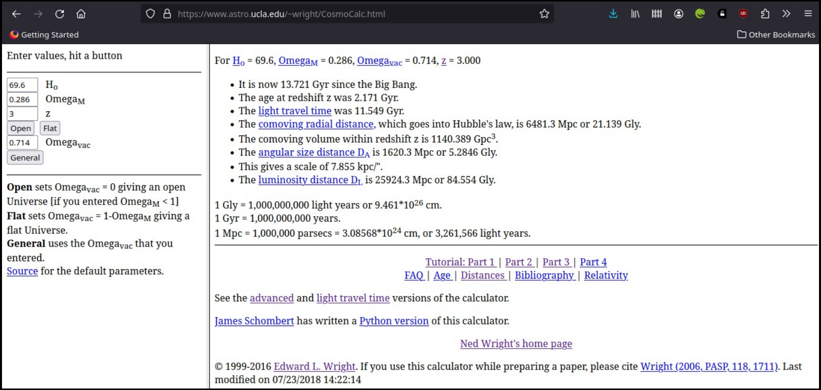
Too many complex equations???? :-(

Use Ned Wright's Cosmology Calculator instead!



https://www.astro.ucla.edu/~wright/CosmoCalc.html

Thank you

References

Lecture Series:

- Cosmology Lectures from Lenny Susskind's Theoretical Minimum (least technical)
- G Srinivasan's Lectures for Astronomical Society of India
- Cosmology Lecture (highly technical)

Online resources:

- Distance measures in cosmology by D. W. Hogg
- Ned Wright's Cosmology Tutorial
- Distances in Cosmology by G. Kauffmann
- GR and Distances by R. Kudritzki
- Ned Wright's Cosmology FAQ